ADO'S THEOREM

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Abstract

We give a proof of the celebrated Ado's Theorem, which states that every finite dimensional Lie Algebra over a field of characteristic 0 admits a faithful finite-dimensional representation. We will also discuss Lie algebra cohomology to prove Levi's Theorem.

Overview of the Proof

The goal of this paper is to give an exposition of the proof of Ado's Theorem, an important "embedding" result often used to simplify the proof of the correspondence between Lie groups and Lie algebras. We will deviate slightly from the standard proof online by introducing Lie Algebra cohomology to prove the Levi Decomposition Theorem, following [1]. For a greater generality and also for the author to learn more about the structural theory.

Theorem 0.1 (Ado's Theorem)**.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field of characteristic* 0*. Then, there exists a faithful finite dimensional representation of $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$ such that the largest nilpotent ideal $\mathfrak{n} \leq \mathfrak{g}$ acts nilpotently on V.*

The idea of the proof is as follows. If $\mathfrak{g}$ is semisimple ($\mathrm{rad}(\mathfrak{g}) = 0$), then it has trivial center, so the natural adjoint action $x \mapsto \mathrm{ad}_x$, which has the kernel equal to the center, gives us a finite dimensional faithful representation. However, not all Lie algebras are nice, in particular some have a nontrivial maximal solvable ideal (called $\mathrm{rad}(\mathfrak{g})$). Thus, we need to tackle the solvable case, and before then we need to tackle the nilpotent case that is less restricted. Finally, there exists a decomposition of $\mathfrak{g}$ to a semidirect product of a solvable and a semisimple component, and then we can patch things together to conclude the proof.

1. Nilpotent and Solvable Lie Algebras

1.1. Universal Enveloping Algebra.

Definition 1.1 (Universal Enveloping Algebra)**.** Let $\mathfrak{g}$ be a Lie algebra. The universal enveloping algebra $U\mathfrak{g}$ is given as the left adjoint of the forgetful functor $\mathsf{AssAlg} \to \mathsf{LieAlg}$. Explicitly, it is the tensor algebra of $\mathfrak{g}$ modulo the ideal generated by the relations $[x, y] - x \otimes y - y \otimes x$.

The Poincare-Birkhoff-Witt theorem asserts that the canonical map $\mathfrak{g} \to U(\mathfrak{g})$ is injective, so $\mathfrak{g}$ acts on $U(\mathfrak{g})$ faithfully to give an isomorphism between $\mathfrak{g}$ and a Lie subalgebra of $\mathrm{End}(U(\mathfrak{g}))$, which is almost always infinite dimensional. Ado's theorem strengthens this result considerably by giving a finite dimensional representation.

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Definition 1.2 (Hopf Algebra)**.** A k-bialgebra $(A, m, \eta, \Delta, \epsilon)$ with multiplication m, comultiplication Δ, unit $\eta : k \to A$ and counit $\epsilon : A \to k$ is called a Hopf algebra if there exists a k-linear function $S : A \to A$, called the antipode or coinverse, such that as a map $A \to A$,

$$m \circ (\mathrm{id} \otimes S) \circ \Delta = m \circ (S \otimes \mathrm{id}) \circ \Delta = \eta \circ \epsilon$$

Proposition 1.3. *$U\mathfrak{g}$ is a Hopf algebra, with the set of primitive elements $\{x \in U\mathfrak{g} : \Delta(x) = x \otimes 1 + 1 \otimes x\}$ equal to $\mathfrak{g}$.*

Proof. The bialgebra structure is given by $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\eta : k \to U\mathfrak{g}$, $\eta(1) = 1$, $\epsilon : U\mathfrak{g} \to k$. $U\mathfrak{g}$ is then a Hopf algebra by $S(x) = -x$ for $x \in \mathfrak{g}$. See [5] for a proof of the rest. □

From now, a $\mathfrak{g}$-module denotes a $U\mathfrak{g}$-module.

1.2. **Semidirect Products.**

Definition 1.4 (Derivation Lie Algebra)**.** Let $\mathfrak{g}$ be a Lie algebra over k. Let $D : \mathfrak{g} \to \mathfrak{g}$ be a k-liner map such that $D([x, y]) = [D(x), y] + [x, D(y)]$. D is called a derivation on $\mathfrak{g}$. The set of derivations, denoted by $\mathrm{Der}(\mathfrak{g})$, is a k-vector space naturally and a Lie algebra with lie bracket $[D, D'] = D \circ D' - D' \circ D$.

Definition 1.5 (Semidirect Products)**.** Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{a} \subset \mathfrak{g}$ be an ideal and $\mathfrak{h}$ be a subalgebra. Suppose the natural map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$ induces a Lie algebra isomorphism $\mathfrak{h} \to \mathfrak{g}/\mathfrak{a}$, i.e. $\mathfrak{g} \simeq \mathfrak{a} \oplus \mathfrak{h}$ as vector spaces. We also have a homomorphism $\theta : \mathfrak{h} \to \mathrm{Der}(\mathfrak{a})$, sending $x \in \mathfrak{h}$ to $\theta(x)(-) = \mathrm{ad}_{-}\, x = [x, -]$. Then, $\mathfrak{g}$ is called the (internal) semidirect product of $\mathfrak{h}$ by $\mathfrak{a}$, and θ is the associated map, and we write $\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{h}$.

Semidirect products let us construct larger Lie algebras from components. Given $\mathfrak{a}, \mathfrak{h}$, and any Lie homomorphism $\mathfrak{h} \to \mathrm{Der}(\mathfrak{a})$, we can construct the (external) semidirect product $\mathfrak{g}$, which is the direct sum of $\mathfrak{a}$ and $\mathfrak{h}$ as a vector space and for $x \in \mathfrak{h}$, we define $[a, x]$ in $\mathfrak{g}$ as $\theta(x)(a)$. This identifies $\theta(x) = \mathrm{ad}_{\mathfrak{g}|_{\mathfrak{a}}}(x)$, and we can check that the Jacobi identity holds. Thus $\mathfrak{g}$ is a Lie algebra and $\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{h}$. This also identifies $\mathfrak{a}$ as an ideal and $\mathfrak{h}$ as a subalgebra of $\mathfrak{g}$. Also, we get a split short exact sequence in LieAlg, $0 \to \mathfrak{a} \to \mathfrak{a} \rtimes \mathfrak{h} \to \mathfrak{h} \to 0$. The splitting is given by $\mathfrak{h} \to \mathfrak{a} \rtimes \mathfrak{h}$, $x \to (0, x)$, which is also a Lie algebra homomorphism.

1.3. **Nilpotency and Solvability.** Let $\mathfrak{g}$ be a Lie algebra.

Definition 1.6 (Nilpotent Lie Algebras)**.** If the upper central series $\mathfrak{g}_n := [\mathfrak{g}, \mathfrak{g}_{n-1}]$ vanishes for some $n \in \mathbb{Z}$, then $\mathfrak{g}$ is called nilpotent.

Definition 1.7 (Solvable Lie Algebras)**.** If the derived series $\mathfrak{g}_n := [\mathfrak{g}_{n-1}, \mathfrak{g}_{n-1}]$ vanishes for some $n \in \mathbb{Z}$, then $\mathfrak{g}$ is called solvable.

Definition 1.8 (Semisimple Lie Algebras, Radical)**.** $\mathfrak{g}$ is called semisimple if $\mathfrak{g}$ does not contain any nontrivial abelian ideal. Equivalently, its largest solvable ideal, called its radical, is zero.

The radical of a Lie algebra always exists and is given as $\sum_{\mathfrak{a} \text{ radical}} \mathfrak{a}$. The sum of two radicals is a radical because $0 \to \mathfrak{a} \to \mathfrak{a} + \mathfrak{b} \to \mathfrak{a}/\mathfrak{a} \cap \mathfrak{b} \to 0$ is exact.

Definition 1.9 (Center)**.** The center of $\mathfrak{g}$ consists of all $x \in \mathfrak{g}$ such that $[x, a] = 0$ for all $a \in \mathfrak{g}$.

In linear algebra, an endomorphism of a nonzero vector space $\sigma : V \to V$ is nilpotent if there exists $n \geq 0$ such that $\sigma^n = 0$. The next theorem explains the connection of nilpotent Lie algebras with this:

Theorem 1.10 (Engel's Theorem)**.** *If $\mathfrak{g}$ is finite dimensional and acts on a nonzero vector space V by nilpotent endomorphisms, then there exists $v \neq 0$, $v \in V$, such that $\mathfrak{g}v = 0$.*

Proof. See [1], Theorem 4.2.2.2. The idea is to construct a codimension 1 subspace $\mathfrak{a}$, and apply induction hypothesis on $\mathfrak{a}$ and $\mathfrak{g}/\mathfrak{a}$ to find a common element that vanishes under the action of $\mathfrak{g}$ to derive a contradiction. □

Corollary 1.11. *Let $\mathfrak{g}$ be finite dimensional. $\mathfrak{g}$ is nilpotent if and only if $(ad_x)^n = 0$ for some n and for all $x \in \mathfrak{g}$, where $ad_x(y) := [x, y]$ is an endomorphism of $\mathfrak{g}$.*

Proof. $\mathfrak{g}$ acts on $\mathfrak{g}$ through the adjoint map by nilpotent endomorphisms, so by Engel's theorem, we can construct a flag $\mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_n = 0$, where each $\mathfrak{g}_i$ has codimension i, n is the dimension of $\mathfrak{g}$, and $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$. This is equivalent to the nilpotency of $\mathfrak{g}$. □

There is an analog of Engel's Theorem for solvable Lie algebras, which we will state but not prove:

Theorem 1.12 (Lie's Theorem)**.** *Let $\mathfrak{g}$ be a finite-dimensional solvable Lie algebra over k of characteristic 0 with algebraic closure $\bar{k}$. Let V be a finite dimensional $\bar{k}$ vector space as well as being acted by $\mathfrak{g}$, then there exists $v \neq 0$, $v \in V$, such that $\mathfrak{g}v = \lambda(\mathfrak{g})v$, where λ is some k-linear mapping of $\mathfrak{g}$ into $\bar{k}$.*

2. Towards Lie Algebra Cohomology

2.1. **Ext.** A general theme in algebra concerns about how to piece small objects together to form larger objects. We have seen an example of this, the semidirect product. They belong to a more general class of study of extensions.

Definition 2.1 (Extensions)**.** The Lie algebra $\hat{\mathfrak{g}}$ is an extension of $\mathfrak{g}$ by $\mathfrak{m}$ if

$$0 \to \mathfrak{m} \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0$$

is exact. Then, we can identify $\mathfrak{m}$ as an ideal in $\hat{\mathfrak{g}}$.

We call the extension abelian if $\mathfrak{m}$ is an abelian ideal of $\mathfrak{g}$.

The extension problem asks which Lie algebras can be constructed as extensions of a Lie algebra by another Lie algebra. This motivates our study of the Ext functor, which is a cohomological invariant that classifies extensions up to equivalences, as we will see below.

Definition 2.2 (Ext Functor)**.** Let U be an associative algebra. Consider the category U-Mod. Fix N a U-module. Then, $F = \mathrm{Hom}(-, N)$ is left exact but not right exact. Abstractly, the Ext functor defined as the right derived functor of F. We write $\mathrm{Ext}^i_U(M) = R^iF(M)$.

The concrete construction is as follows: the Let M be a U-module. Take the free resolution of M (which always exists) and remove the term M, we get

$$P_\bullet = 0 \leftarrow P^0 \leftarrow P^1 \leftarrow \cdots .$$

Then, the i-th Ext functor is the i-th cohomology of the above sequence after applying F:

$$\mathrm{Ext}^i_U(M,N) = H^i(F(P_\bullet)).$$

Then, $\mathrm{Ext}^0_U(M,N) = \mathrm{Hom}(M,N)$.

We need to check that Ext does not depend on the choice of injective resolutions. This can be shown by lifting the map $M \to M$ to a chain homotopy, which induces an isomorphism on the homology. For the details see [3].

Definition 2.3 (Lie Algebra Cohomology)**.** Let $\mathfrak{g}$ be a Lie algebra over k. The abelian cohomology of $\mathfrak{g}$ with coefficients in the left $\mathfrak{g}$-module M is $H^\bullet_{\mathrm{Lie}}(\mathfrak{g},M) = \mathrm{Ext}^\bullet_{U\mathfrak{g}}(k,M)$, with k as the trivial g-module given by $u \cdot 1 = 0$ for $u \in U\mathfrak{g}$.

The original definition of the Lie algebra cohomology was via a $U\mathfrak{g}$-module chain complex by C. Chevalley and S. Eilenberg in 1948. We will use this version to compute the cohomology groups. We will state the general definition for completeness, but here we will only need Ext^1 and Ext^2, so we will not discuss the general theory. The reader can read [3] if they are interested.

Definition 2.4 (Wedge Product)**.** Let V be a vector space over a field k. The wedge product $\bigwedge V$ is defined as the quotient algebra of the tensor algebra by the ideal I generated by the form $x \otimes x$. We write $x \wedge y := x \otimes y \pmod I$.

Definition 2.5 (Exterior Product)**.** The k-th exterior power $\bigwedge^k V$ is the vector subspace of $\bigwedge V$ spanned by elements of the form $x_1 \wedge \cdots \wedge x_k$.

Definition 2.6 (Chevalley-Eilenberg Cochain Complex)**.** Let $\mathfrak{g}$ be a Lie algebra over k. Then, k has a free $U\mathfrak{g}$-resolution given by

$$\cdots \to U(\mathfrak{g}) \otimes \bigwedge^2 \mathfrak{g} \xrightarrow{d_2} U(\mathfrak{g}) \otimes \mathfrak{g} \xrightarrow{d_1} U(\mathfrak{g}) \xrightarrow{\epsilon} k \to 0.$$

The Chevalley-Eilenberg cochain complex is given by

$$(\mathrm{CE}(\mathfrak{g},M), \delta^\bullet) := \mathrm{Hom}_{\mathfrak{g}}(\bigwedge^\bullet \mathfrak{g}, M)$$

The differential $\delta^k : \mathrm{Hom}_{\mathfrak{g}}(\bigwedge^{k-1} \mathfrak{g}, M) \to \mathrm{Hom}_{\mathfrak{g}}(\bigwedge^k \mathfrak{g}, M)$ is given by

$$\begin{aligned}\delta^k\theta\left(x_1 \wedge \cdots \wedge x_k\right) = &\sum_i (-1)^{i-1} x_i \theta\left(x_1 \wedge \cdots \hat{x}_i \cdots \wedge x_k\right) \\ &- \sum_{i<j} (-1)^{i-j+1} \theta\left([x_i,x_j] \wedge x_1 \wedge \cdots \hat{x}_i \cdots \hat{x}_j \cdots \wedge x_k\right).\end{aligned}$$

It turns out that magically, the homology of this cochain complex agrees with the Ext functor!

Fact 2.7. $H^\bullet_{\mathrm{Lie}}(\mathfrak{g},M) = \mathrm{Ext}^\bullet(k,M)$ is the homology of the Chevalley-Eilenberg cochain complex.

The proof of this is in ([3], Theorem 7.7.2), which involves Hochschild-Serre spectral sequences so it is outside the scope of our discussion.

Let us specialize to the first three terms. To shorten the notation, let $V^i = \mathrm{Hom}_{\mathfrak{g}}(\bigwedge^i \mathfrak{g}, M)$. If $\theta \in V^1$, then the formula above reads

$$\delta^2(\theta)(x_1 \wedge x_2) = x_1\theta(x_2) - x_2\theta(x_1) - \theta([x_1, x_2]).$$

$\delta^2 : V^1 \to V^2$ is a linear map, and thus the kernel is a subspace. Denote this by C^1, the space of 1-cocycles. Notice that if $\delta^2(\theta) = 0$, then $\theta \in \mathrm{Der}(\mathfrak{g}, M)$, so $C^1 = \mathrm{Der}(\mathfrak{g}, M)$.

When $k = 1$, $\theta \in V^0$, we have $\delta^1(\theta)(x_1) = x_1 a$ for some $a \in M$. Denote this as θ_a. The set of all θ_a is thus a subspace of C^1, which we will call B^1, the space of 1-coboundaries. It is also called the space of inner derivations $\mathrm{Ider}(\mathfrak{g}, M)$. Thus,

$$H^1_{\mathrm{Lie}}(\mathfrak{g}, M) = \frac{\mathrm{Der}(\mathfrak{g}, M)}{\mathrm{Ider}(\mathfrak{g}, M)}.$$

The first Ext group measures how far the derivation subspace is being inner.

2.2. **Ext of Semisimple Lie Algebras.** In this quick disgression, we will look at the Ext functors of semisimple Lie algebras. Attempting to classify the extensions of all Lie algebra turns out to be hopeless. However, all hope is not lost: a good deal about semisimple Lie algebras is known. As the first step towards understanding them, we prove that their Lie algebra cohomology vanishes, also known as Whitehead's Theorem.

Proposition 2.8. $\mathrm{Ext}^i(M, N) \simeq \mathrm{Ext}^i(\mathrm{Hom}(N, M), k) \simeq \mathrm{Ext}^i(k, \mathrm{Hom}(N, M))$.

Proof. ([1], Corollary 4.4.3.2.) □

Theorem 2.9. *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field k of characteristic 0, and let M be a finite dimensional g-module. Then,* $\mathrm{Ext}^1(M, N) = 0$.

Proof. ([6], Theorem 3.12.1.) In view of the previous proposition and fact 2.7, $\mathrm{Ext}^1(M, N) = H^1_{\mathrm{Lie}}(\mathfrak{g}, M)$. For $x \in \mathfrak{g}$, let $\pi(x)$ be the endomorphism of V^1 defined by

$$(\pi(x)\psi)(y) = -\psi([x, y]) + x\psi(y)$$

for $\psi \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, M)$. Then, $\pi : \mathfrak{g} \to \mathrm{End}(V^1) = \mathrm{End}(\mathrm{Hom}(\mathfrak{g}, k)) \simeq \mathrm{End}(\mathfrak{g})$ is a representation of $\mathfrak{g}$. We also have $-\psi([x, y]) + x\psi(y) = y\psi(x) = \theta_{\psi(x)}(y)$, so $\pi(x)\psi = \theta_{\psi(x)}$. In other words, $\pi(x)$ maps C^1 into B^1. Let $\pi'(X) = \pi(X)|_{C^1}$. Thus, we have the decomposition

$$C^1 = C^1_n \oplus C^1_r,$$

where $C^1_r = \sum_{x\in\mathfrak{g}} \pi'(x)(C^1) \subset B^1$, and $C^1_n = \{v \in C^1 : \pi'(x)v = 0 \, \forall x \in \mathfrak{g}\}$. Since $B^1 \subset C^1$, it suffices to show that $C^1_n = 0$. Let $\psi \in C^1_n$, then

$$\pi'(x)(\psi)(y) = -\psi([x, y]) + x\psi(y) = 0$$

for all $x, y \in \mathfrak{g}$. Also, since $d\psi = 0$, we have $x\psi(y) - y\psi(x) - \psi([x, y]) = 0$, so $y\psi(x) = 0$. Since x, y are arbitrary, $x\psi(y) = 0$. Therefore, we have shown

$$\psi([\mathfrak{g}, \mathfrak{g}]) = 0.$$

Finally, we claim that for semisimple Lie algebras, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Indeed, if not, then $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ will also be a semisimple Lie algebra (the proof requires Cartan's theory of Killing forms), but $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is a nontrivial abelian Lie algebra, which is impossible. Thus, $\psi(\mathfrak{g}) = 0$, so $\psi = 0$. This completes the proof that $C^1 = B^1$, so $H^1(\mathfrak{g}, k) = \mathrm{Ext}^1(M, N) = 0$. □

Corollary 2.10 (Whitehead's Theorem)**.** *If $\mathfrak{g}$ is a semisimple Lie algebra over characteristic zero, and M and N are finite dimensional non-isomorphic simple $\mathfrak{g}$-modules, then $\mathrm{Ext}^i(M, N)$ vanishes for all i.*

Proof. This follows from a general fact in homological algebra that if Ext^1 vanishes, then all the higher Ext^ks vanish. See [3], Exercise 2.5.2, for example. ☐

The following theorem is a corollary of Theorem 2.6 and is one of the most fundamental theorems in the theory of semisimple Lie algebras. It also justifies the usage of the term "semisimple" as semisimple in group theory or representation theory is often referred to as a direct sum of simple objects. The proof is omitted.

Definition 2.11 (Completely Reducible)**.** An object in an abelian category is simple if it has no non-zero proper subobjects. An object is completely reducible if it is a direct sum of simple objects.

Theorem 2.12 (Weyl's Complete Reducibility Theorem)**.** *Every finite dimensional representation of a semisimple Lie algebra over characteristic zero is completely reducible.*

2.3. **Levi Decomposition Theorem.** We now explain how the Ext functor classifies extensions of Lie algebras.

Theorem 2.13 (Ext^2 Classifies Extensions)**.** $\mathrm{Ext}^2(\mathfrak{g}, \mathfrak{m})$ *classifies extensions* $0 \to \mathfrak{m} \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0$ *up to isomorphism. The element* $0 \in \mathrm{Ext}^2$ *corresponds to the semidirect product* $\hat{\mathfrak{g}} = \mathfrak{m} \rtimes \mathfrak{g}$.

Proof. We can define algebraically a correspondence between isomorphism classes of extensions of $\mathfrak{g}$ by m with elements in Ext^2.

Since all the terms above are free as modules over k, we can always pick a k-linear splitting $\sigma : \mathfrak{g} \to \hat{\mathfrak{g}}$. Let f be the function

$$f(x, y) : [\sigma(x), \sigma(y)] - \sigma([x, y])$$

f is alternating and bilinear, so $f \in V^2$.

In fact, f is in C^2, a 2-cocycle. To see this, we use Definition 2.6 to calculate the differential:

$$\begin{aligned}(\delta^3 f)(x, y, z) = xf(y, z) + yf(x, z) + zf(x, y) - \\ (f([x, y], z) + f([y, z], x) + f([z, x], y))\end{aligned}$$

The right hand side, if expanded out, is equivalent to the Jacobi identity of the Lie bracket of $\hat{\mathfrak{g}}$. Thus, $\delta^3 f = 0$. The choice of the homology class of f does not depend on the choice of the linear splitting. We can check that if σ' is another splitting and f' is defined similarly as f, then $f - f'$ is a 2-coboundary, so $[f] = [f']$ in Ext^2. Thus, we get a map from the extension classes to elements in $\mathrm{Ext}^2(\mathfrak{g}, \mathfrak{m})$.

The proof that this is a 1-1 correspondence is skipped. ☐

Since $H^2(\mathfrak{g}, k) = 0$ for $\mathfrak{g}$ semisimple by Theorem 2.9, it follows that the abelian extensions of $\mathfrak{g}$ are precisely semidirect products. This leads us to an important classification of finite dimensional Lie algebras:

Theorem 2.14 (Levi Decomposition Theorem)**.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over characteristic* 0*, and let* $\mathfrak{r} = \mathrm{rad}(\mathfrak{g})$*. Then $\mathfrak{g}$ has a Levi decomposition: $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ where $\mathfrak{s}$ is semisimple.*

Proof. If $\mathfrak{r} = 0$ then $\mathfrak{g}$ is already semisimple, so assume $\mathfrak{r} \neq 0$. Then, $\mathfrak{s} = \mathfrak{g}/\mathfrak{r}$ is semisimple.

If $[\mathfrak{r}, \mathfrak{r}] = 0$, then $\mathfrak{r}$ is abelian, so every choice of splitting $\mathfrak{s} \to \mathfrak{g}$ correspond to the zero element in Ext^2, so $\mathfrak{g}$ has a Levi decomposition.

If $[\mathfrak{r}, \mathfrak{r}] \neq 0$, let $\mathfrak{h} = \mathfrak{g}/[\mathfrak{r}, \mathfrak{r}]$ and $\pi : \mathfrak{g} \to \mathfrak{h}$ be the natural map. Then, $\pi(\mathfrak{r})$ is a radical of $\mathfrak{g}'$, and we have a Levi decomposition by induction $\mathfrak{g}' = \pi(\mathfrak{r}) \rtimes \mathfrak{m}'$. Let $\mathfrak{m}_0 = \pi^{-1}(\mathfrak{m})$, then $\mathfrak{g} = \mathfrak{r} + \mathfrak{m}_0$ and $[\mathfrak{r}, \mathfrak{r}] = \mathfrak{r} \cap \mathfrak{m}_0$.

Since $[\mathfrak{r}, \mathfrak{r}]$ is a solvable ideal of $\mathfrak{m}_0$, and $\mathfrak{m}_0/[\mathfrak{r}, \mathfrak{r}]$ is isomorphic to $\mathfrak{m}'$, so it is semisimple, so $[\mathfrak{r}, \mathfrak{r}] = \mathrm{rad}(\mathfrak{m}_0)$. By induction hypothesis, we can find $\mathfrak{m}$ an ideal of $\mathfrak{m}_0$ such that $\mathfrak{m}_0 = [\mathfrak{r}, \mathfrak{r}] + \mathfrak{m}$. We have $\mathfrak{g} = \mathfrak{r} + \mathfrak{m}$ and $\mathfrak{r} \cap \mathfrak{m} = 0$, so $\mathfrak{g}$ has a Levi decomposition, as required. □

3. Zassenhaus's Extension Lemma

The following theorem follows ([2] Ch.6), which in turn is based on ([4], I, §.7).

Let $\mathfrak{h}$ and $\mathfrak{a}$ be finite dimensional Lie algebras such that $\mathfrak{h}$ acts on $\mathfrak{a}$ by derivations, then we can form $\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{h}$. If we have a $\mathfrak{a}$-module V, can we extend it to a $\mathfrak{g}$-module W while keeping control of the nilpotent components?

Definition 3.1 (Nilpotency Ideal)**.** Let $\mathfrak{g}$ act on V to give a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ The largest ideal of $\mathfrak{g}$ that acts nilpotently through ρ is called the nilpotency ideal $n_\rho(\mathfrak{g})$.

For example, one can show that if $\mathfrak{g}$ is a finite dimensional Lie algebra, then by corollary 1.11 the nilpotency ideal $\mathfrak{n}$ of the adjoint action $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is the largest nilpotent ideal of $\mathfrak{g}$.

Theorem 3.2 (Zassenhaus)**.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over k. A finite dimensional representation ρ' of a Lie subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ extends to a finite dimensional representation ρ of $\mathfrak{g}$ such that $n_\rho(\mathfrak{g}) \supset n_{\rho'}(\mathfrak{g}')$ if $\mathfrak{g}'$ is an ideal in $\mathfrak{g}$ and there exists a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{g}'] \subset n_{\rho'}(\mathfrak{g}')$.*

Proof. Let $\rho' : \mathfrak{g}' \to \mathfrak{gl}(V')$ be a finite dimensional representation of $\mathfrak{g}'$. By Poincare-Birkhoff-Witt, we can pick a totally ordered basis on $\mathfrak{g}$ that extends to a basis on $U(\mathfrak{g})$, thus we get a representation $\sigma : U(\mathfrak{g}') \to \mathfrak{gl}(V')$. Let I be the kernel of σ. As a vector space, we know that I has finite codimension.

Since $\mathfrak{g}'$ is an ideal, $U(\mathfrak{g}') \subset U(\mathfrak{g})$ is a $\mathfrak{g}$-module, so the dual space $U(\mathfrak{g}')^\vee = \mathrm{Hom}(U(\mathfrak{g}'), k)$ is a $\mathfrak{g}$-module naturally. To make progress, we construct a $\mathfrak{g}$-module that "records" the data of the representation in the dual space: fix a basis for V', then for all $x \in U(\mathfrak{g}')$ we get $\rho(x) = (\rho_{ij}(x))$, where $\rho_{ij} : U(\mathfrak{g}') \to k$ gives the ij-th entry of the matrix of $\rho(x)$. The vector space $C(\rho')$ spanned by ρ_{ij} thus is a subspace of $U(\mathfrak{g}')^\vee$. Let S be the $\mathfrak{g}$-submodule generated by elements of $C(\rho')$. We claim that S is finite dimensional.

With that, since V' embeds into S^n for $n = \dim V'$ via mapping the basis vector v_i of V' into $\sum_{j=1}^{\dim V'} \rho_{ij}$, and the latter is a $\mathfrak{g}$-module which is finite dimensional, so we get a finite dimensional representation $\rho : \mathfrak{g} \to \mathfrak{gl}(S^n)$. It remains to prove the claim and control the nilpotency ideal of ρ'.

Pick a Jordan-Holder series for V', i.e. a filtration $V' = V'_0 \supset V'_1 \supset \cdots \supset V'_d = 0$ such that V'_{i-1}/V'_i is simple. Then, ρ is invariant on each V'_i, so we get the induced representations $\sigma_i : U(\mathfrak{g}') \to V'_{i-1}/V'_i$ for $i = 1, \cdots, d$. Let $I' = \bigcap \ker \sigma_i$. We claim then that $I' \cap \mathfrak{g}' = n_{\rho'}(\mathfrak{g}')$, so we can understand $n_{\rho'}(\mathfrak{g}')$. If $x \in \bigcap \ker \sigma_k \cap \mathfrak{g}'$, then $\sigma_k(x) = 0$ on V'_{k-1}/V'_k for all k, so if $\rho'(x)(v) \in V'_{k-1}$, then $(\rho'(x))^2(v) \in V'_k$, so applying $\rho'(x)$ at most d times gives 0, so $x \in n_{\rho'}(\mathfrak{g}')$. Reversing the argument gives the other inclusion.

What we have also shown above is that I'^d acts as 0 in ρ', so $I'^d \subset I$, so I'^d has finite codimension.

Consider the derivation $U(\mathfrak{g}') \to U(\mathfrak{g}')$ as $u \mapsto [x, u] = xu - ux$ for $x \in \mathfrak{h}$. Then, the image of $\mathfrak{g}'$ is in $[\mathfrak{h}, \mathfrak{g}']$, and by assumption $[\mathfrak{h}, \mathfrak{g}'] \subset n_{\rho'}(\mathfrak{g}') \subset I'$. Therefore, the image of $U(\mathfrak{g}')$ is in I'. Since $I' \subset U(\mathfrak{g}')$, the image of I'^d is in I'^d, so this means that $x \cdot I'^d \subset I'^d$, so I'^d is actually a $\mathfrak{g}$-module. The orthogonal complement of I'^d in $U(\mathfrak{g}')$ as a vector space is finite dimensional. Moreover, if we dualize this orthogonal complement, we see that it contains $C(\rho')$, and therefore S. Thus, S is finite dimensional.

$\rho(x)$ is nilpotent for $x \in I' \cap \mathfrak{g}'$ by the definition of I', so the nilpotency ideal $n_\rho(\mathfrak{g})$ contains $I' \cap \mathfrak{g}' = n_{\rho'}(\mathfrak{g}')$. This completes the proof. □

Lemma 3.3. *Let $\mathfrak{r}$ be a solvable Lie algebra over characteristic 0 , and let $\mathfrak{n}$ be its largest nilpotent ideal. Then every derivation of $\mathfrak{r}$ has image in $\mathfrak{n}$. In particular, if $\mathfrak{r}$ is an ideal of some larger Lie algebra $\mathfrak{g}$, then $[\mathfrak{g}, \mathfrak{r}] \subseteq \mathfrak{n}$.*

Proof. The second part follows from the first part since $[g, -]$ is a derivation of $\mathfrak{r}$ for $g \in \mathfrak{g}$. Let D be a derivation of $\mathfrak{r}$, consider the semidirect product $\mathfrak{g} = \mathfrak{r} \rtimes k$ given by $\theta : k \to \mathrm{Der}(\mathfrak{r})$, $1 \mapsto D$. Then,

$$[(x, t), (x', t')] = (tD(x') - t'D(x) + [x, x'], 0).$$

Since $\mathfrak{g}/\mathfrak{r} = k$ and $\mathfrak{r}$ are solvable, so is $\mathfrak{g}$. Embed $\mathfrak{r}$ into $\mathfrak{g}$ as $r \mapsto (r, 0)$. Then $\mathfrak{r}$ is a solvable ideal of $\mathfrak{g}$. Therefore,

$$D(\mathfrak{r}) = [(0, 1), (\mathfrak{r}, 0)] \subset \mathfrak{r} \cap [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{r} \cap \mathfrak{n}_{\mathfrak{g}} \subset \mathfrak{n}.$$

□

4. Ado's Theorem, characteristic 0

Proof of Theorem 1. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Induct on the dimension of $\mathfrak{g}$: if $\mathfrak{g} = 0$, the theorem is trivial. For the induction step:

($\mathfrak{g}$ is nilpotent.) By the same argument in proving Engel's theorem, we can find a codimensional 1 Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$. Pick $x \notin \mathfrak{a}$ such that $\mathfrak{h} = \langle x \rangle$, then $\mathfrak{h}$ is a subalgebra, and we get $\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{h}$. By induction, there exists a finite dimensional vector space V' such that $\mathfrak{a}$ acts faithfully and nilpotently.

By Theorem 3.2, the representation $\rho' : \mathfrak{a} \to \mathfrak{gl}(V')$ extends to a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(W)$ such that $\mathfrak{g}$ acts nilpotently on W ($\mathfrak{g}$ is the largest nilpotent ideal of $\mathfrak{g}$). Also, since $\mathfrak{g}/\mathfrak{a}$ is dimension 1, we can find a nilpotent $\mathfrak{g}/\mathfrak{a}$-module W' such that $\mathfrak{a}$ is in the kernel in its associated representation. Thus, $W \oplus W'$ gives a nilpotent faithful representation of $\mathfrak{g}$.

($\mathfrak{g}$ is solvable but not nilpotent.) Let $\mathfrak{n}$ denote the largest nilpotent ideal of $\mathfrak{g}$. By Lie's Theorem, we can pick an ideal $\mathfrak{a}$ of $\mathfrak{g}$ of codimension 1 such that $\mathfrak{n} \subset \mathfrak{a}$ and $\mathfrak{h} = kx$ ($x \notin \mathfrak{a}$) such that $\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{h}$. Thus, we get a finite dimensional vector space V' by induction of which $\mathfrak{a}$ acts faithfully via representation ρ', and $n_{\rho'}(\mathfrak{a}) \supset \mathfrak{n}$. Then, $[\mathfrak{h}, \mathfrak{a}]$ acts on V' nilpotently. Moreover, by Lemma 3.3,

$[h, \mathfrak{a}] \subset n_{\rho'}(\mathfrak{a})$, so by Theorem 3.2, we get an extension of representation $\rho : \mathfrak{g} \to \mathfrak{gl}(W)$ such that $\mathfrak{n} \subset n_\rho(\mathfrak{g})$ acts nilpotently on W. As above, we can also pick a faithful representation W' of $\mathfrak{g}/\mathfrak{a}$, so $\mathfrak{g}$ acts on $W \oplus W'$ faithfully.

(General case.) By Levi's Theorem, there is a decomposition $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$, where $\mathfrak{r} = \mathrm{rad}(\mathfrak{r})$ and $\mathfrak{s}$ is semisimple. By the previous case, there is a faithful finite-dimensional representation V of $\mathfrak{r}$ such that $[\mathfrak{h}, \mathfrak{r}] \leq \mathfrak{n}$ (because every derivation of $\mathfrak{r}$ has image in $\mathfrak{n}$), so by Theorem 3.2, we can extend to a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(W)$ such that $\mathfrak{n}$ acts on W nilpotently and faithfully. Also, we know that ad acts on $\mathfrak{s}$ to give a faithful representation, so let $W' = \mathfrak{s}$, then $\mathfrak{g}$ acts on $W \oplus W'$ faithfully, proving Theorem 1. □

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