MODULAR FORMS AND THE KODAIRA-SPENCER ISOMORPHISM

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Abstract. This expository article is my final project for Math 205B at UCLA on Modular Forms for Winter 2026, taught by Romyar Sharifi. We explain Katz's algebraic perspective of modular forms of level 1 as sections of tensor powers of the Hodge bundle $\underline{\omega}$. We give a brief survey of algebraic de Rham cohomology to get an isomorphism of Kodaira-Spencer $\Omega^1 \simeq \underline{\omega}^{\otimes 2}$ in the complex case.

1. Modular forms of Level 1

A classical modular form of weight k and level 1 is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ satisfying $f(\gamma\tau) = j(\gamma, \tau)^k f(\tau)$ for all $\gamma \in SL_2(\mathbb{Z})$ and $\tau \in \mathcal{H}$ which is holomorphic at infinity. Our object of interest is Katz's recast of modular forms [7] in 1973, which permits to generalize modular forms to over arbitrary rings and to define p-adic modular forms.

The manifestation in the $\mathbb{C}$ case is as follows. As $\mathcal{H}$ is the coarse moduli space of elliptic curves over $\mathbb{C}$, we may view $f(\tau)$ as a function $F(\mathbb{Z}\tau + \mathbb{Z})$ taking values in elliptic curves, if we express an elliptic curve as a lattice $E = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$. However, it is not clear how we can generalize this to elliptic curves over arbitrary rings. For one, we actually made a very specific choice of representation of the elliptic curve here by picking $\{1, \tau\}$ as a basis. There can be infinitely many choices up to homothety by scaling the lattice with $\lambda \in \mathbb{C}^\times$. To rigidify the choice, we specify a nowhere vanishing holomorphic differential $\omega \in \Omega^1_{E/\mathbb{C}}$. It has to be nowhere vanishing because heruistically, otherwise we cannot measure the size of the lattice uniformly. Then we have the classical correspondence:

$$\{\text{Lattices } \Lambda \subset \mathbb{C}\} \mapsto \{y^2 = 4x^3 - g_2 x - g_3\}$$
$$\{\text{Translation invariant 1-form } \omega = dz\} \mapsto \{\omega = \frac{dx}{y}\},$$

where $\mathbb{C}/\Lambda \ni z \mapsto (x = \wp(z, \Lambda), y = \wp'(z, \Lambda))$, $g_2 = 60 \sum_{0 \neq \lambda \in \Lambda} \lambda^{-4}$, $g_3 = 140 \sum_{0 \neq \lambda \in \Lambda} \lambda^{-6}$. Conversely, given an elliptic curve $E/\mathbb{C}$ defined as an affine variety with a non-zero everywhere holomorphic differential form ω on E, we have

$$\Lambda(E, \omega) = \{\int_\gamma \omega \mid \gamma \in H_1(E, \mathbb{Z}) \simeq \mathbb{Z} \times \mathbb{Z}\} \leftarrow\!\shortmid \{E/\mathbb{C}, \omega\}$$

So now, we may associate to a weight k level 1 modular form f a uniquely determined function

$$F(E/\mathbb{C}, \omega) := \omega_2^{-k} f(\omega_1/\omega_2), \text{ if } \Lambda(E, \omega) = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \text{ and } \operatorname{Im}(\omega_1/\omega_2) > 0.$$

Scaling the differential form by $\lambda \in \mathbb{C}^\times$ would give $F(E, \lambda\omega) = \lambda^{-k} F(E, \omega)$, because the right hand side is a weight $-k$ homogeneous polynomial.

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In the first section, we generalize this to modular forms taking values in elliptic curves over any base scheme. Due to time limitations, we will not use the viewpoint of stacks.

Definition 1.1 (Family of elliptic curves over a scheme)**.** Let S be a scheme. A family of elliptic curves over S is a smooth proper morphism $\pi : X \to S$, whose geometric fibres are elliptic curves, and a section $e : S \to X$.

Example 1. Denote $\mathcal{H}$ as the upper-half plane. Consider the action of $\mathbb{Z}^2$ on $\mathbb{C} \times \mathcal{H}$ by $(m, n) \cdot (z, \tau) = (z + m + n\tau, \tau)$. This action is free and proper, so its orbit space, denoted $\mathbb{X}$, is a smooth 2-manifold. The projection into second component gives a family of elliptic curves over $\mathbb{C}$

$$\pi : \mathbb{X} \to \mathcal{H}$$

It is clear that the fibres $\mathbb{X}_\tau$ are complex elliptic curves $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. The section $e : \mathbb{H} \to \mathbb{X}$ is the zero section. In fact, it can be shown that every complex family of elliptic curves is a pullback of π (in the language of stacks, π is the universal family over $\mathcal{H}$.

Example 2 (Tate Family for Complex Elliptic Curves)**.** . The pair $(\mathbb{C}/(2\pi i\mathbb{Z} + 2\pi i\tau\mathbb{Z}), 2\pi i dz)$ is isomorphic to $(\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), dz)$, and by a change of variables $z \mapsto t = e^{2\pi i z}$, it is equivalent to the pair $(\mathbb{C}^*/q^{\mathbb{Z}}, dt/t)$, where $q = e^{2\pi i\tau}$. We compute the equation for $(\mathbb{C}/(2\pi i\mathbb{Z} + 2\pi i\tau\mathbb{Z}), 2\pi i dz)$ under the correspondence to get

$$y^2 = 4x^3 - \frac{E_4}{12}x + \frac{E_6}{216}, \omega = \frac{dx}{y}.$$

As E_4 and E_6 are in $\mathbb{Z}[[q]]$, this curve has coefficients in $\mathbb{Z}[\frac{1}{2}, \frac{1}{3}][[q]] = \mathbb{Z}[\frac{1}{6}][[q]]$. Applying the change of coordinates $x \mapsto x + \frac{1}{12}$, $y \mapsto x + 2y$, we get

$$y^2 + xy = x^3 + B(q)x + C(q), \omega = \frac{dx}{x + 2y},$$

where $B(q) = -5\sum_{n\geq 1} \sigma_3(n)q^n$, $C(q) = \sum_{n\geq 1}(\frac{-5\sigma_3(n) - 7\sigma_5(n)}{12})q^n$, $\sigma_k(n) = \sum_{d|n, d\geq 1} d^k$. They have coefficients over $\mathbb{Z}[[q]]$. But for each of them to be smooth, we need that the discriminant

$$\Delta = q\prod_{n=1}^{\infty}(1 - q^n)^{24}$$

to be invertible; so each elliptic curve is defined over $\mathbb{Z}[[q]](\frac{1}{q}) = \mathbb{Z}((q))$. This family of elliptic curves forms a morphism from a closed subscheme of $\mathbb{P}^2_S$ to $S = \operatorname{Spec} \mathbb{Z}((q))$, called the Tate family Tate(q). Over each fibre, the canonical differential is defined as $\frac{dx}{x+2y}$.

Next we give the definition of the relative sheaf of differentials. We restrict our attention to a family of elliptic curves $\pi : X \to S$ when the base scheme $S = \operatorname{Spec}(R)$ is affine (no extra assumption on R except it is commutative unital). The most general definition is in ([5], II.8).

Definition 1.2 (Relative Sheaf of Differentials (Local Version))**.** With notation as above. We start with an affine open $U = \operatorname{Spec}(A)$ of X and describe $\Gamma(U, \Omega_{X/S})$.

As π is proper, it is of finite type, so A is a R-algebra. We define $\Gamma(U, \Omega_{X/S})$ as the sheaf of the A-module of $\Omega_{A/R}$, where

$\Omega_{A/R}$ is the free B-module $\langle da : a \in A\rangle$ modulo the rules: for all $a, a' \in A, r \in R$,

$$d(a+a') = da + da', d(aa') = ada' + a'da\,, dr = 0.$$

One can check that they patch together to form a sheaf on X. Additionally, smoothness of π implies that $\Omega_{X/S}$ is an invertible sheaf of $\mathrm{rank}_x(\Omega_{X/S}) = \dim_x(X_{\pi(x)}) = 1$ by ([8], 2:29.34.12), so $\Omega_{X/S}$ is a line bundle on X.

One question might be to classify when does the relative sheaf of differentials for X/S have global sections, and is left to the interested reader.

Example 3 (Tate family)**.** With the notation as in Example 2, the canonical differential $\frac{dx}{x+2y}$ over each fibre patch together to a nowhere vanishing global section $\Gamma(X, \Omega_{X/S})$. This trivializes the line bundle $\Omega_{X/S}$, so we get a pair denoted $(\mathrm{Tate}(q), \omega_{can})$.

Definition 1.3 (Hodge Bundle)**.** Let $\pi : X \to S$ be a family of elliptic curves. The Hodge bundle associate to this family is

$$\underline{\omega}_{X/S} := \pi_*(\Omega_{X/S}).$$

We have come to the position to define the analog of modular forms from Katz's 1973 paper [7]. In this article, we will call Katz's definition of modular forms *premodular forms* for it does not include the holomorphy at infinity condition.

Definition 1.4 (Premodular forms)**.** A premodular form of weight $k \in \mathbb{Z}$ and level one is a rule f which assigns to any family of elliptic curves $\pi : X \to S$ (S is any scheme) a section $f(X/S)$ of $(\underline{\omega}_{X/S})^{\otimes k}$ over S such that the following two conditions are satisfied.

(1) $f(X/S)$ depends only on the S-isomorphism class of the elliptic curve X/S.
(2) The formation of $f(X/S)$ commutes with arbitrary change of base $g : S' \to S$ (meaning that $f(X_{S'}/S') = g^* f(X/S)$).

Definition 1.5. Suppose we only consider schemes S over a base ring R_0, and change of base by R_0-morphisms, then we define a premodular form of weight $k \in \mathbb{Z}$ and level one over R_0 similarly.

Proposition 1. When $S = \mathrm{Spec}(R)$ (R is always commutative unital), the premodular form f admits the following concrete description.

(1) f is a rule which assigns to every pair $(X/R, \omega)$ where ω is a nowhere vanishing section of $\Omega_{X/S}$, an element $f(X/R, \omega) \in R$.
(2) $f(X/R, \omega)$ only depends on the R-isomorphism class of $(X/R, \omega)$.
(3) For any $\lambda \in R^\times$, $f(E, \lambda\omega) = \lambda^{-k} f(E, \omega)$.
(4) If $g : R \to R'$ induces a change of basis, then $f(E_{R'}/R', \omega_{R'}) = g(f(E/R, \omega)) \in R'$.

Proof. $\Omega_{X/S}$ is locally free of rank 1, and as the proposition is a local question, we may assume that $\Gamma(S, \underline{\omega})$ is a free R-module with basis ω. Define $f(X/R, \omega)$ by the characterizing property $f(X/\mathrm{Spec}(R)) = f(X/R, \omega) \cdot \omega^{\otimes k}$. Then it is clear that $f(X/R, \omega)$ satisfies (1)-(4). □

When $S = \mathrm{Spec}(\mathbb{C})$, we get back the classical level 1 (pre)modular forms of weight k described in the introduction. (Of course, under this definition, one has to give a rule to family of elliptic curves over *any* base scheme, not just over $\mathbb{C}$!)

Definition 1.6 (Holomorphy at infinity)**.** Suppose f is a premodular form of weight k and level 1 over R_0. The assignment $F(\mathrm{Tate}(q), \omega_{can})_{R_0}$ Tate curve precisely represents the q-expansion of modular form, where $\mathrm{Tate}(q)$ is defined over $\mathrm{Spec}(\mathbb{Z}((q))\otimes_{\mathbb{Z}} R_0)$ now. We say that f is holomorphic at infinity if $f(\mathrm{Tate}(q), \omega_{can})_{R_0} \in \mathbb{Z}((q)) \otimes_{\mathbb{Z}} R_0$ is actually in $\mathbb{Z}[[q]] \otimes_{\mathbb{Z}} R_0$.

Definition 1.7 (Modular form of level 1)**.** With f as above, if it is holomorphic at infinity, then it is called a modular form.

2. Algebraic de Rham Cohomology

For the second half of the paper, we shall study the Hodge bundle ω in much more detail.

Definition 2.1 (de Rham complex, ([8] 0FK4))**.** Let $\pi : X \to S$ be a family of elliptic curves. Since $\Omega_{X/S}$ is quasi-coherent on X, it is an $\pi^{-1}\mathcal{O}_S$-module. Let $T^n(\Omega_{X/S}) = \Omega_{X/S}^{\otimes k}$, where $T^0 = \pi^{-1}\mathcal{O}_S =: \mathcal{O}_{X/S}$. The *tensor algebra* of $\Omega_{X/S}$ is the direct sum

$$T(\Omega_{X/S}) = \bigoplus_{n\geq 0} T^n(\Omega_{X/S}).$$

The *exterior alegrba* $\bigwedge(\Omega_{X/S})$ is the quotient of $T(\Omega_{X/S})$ by the two-sided ideal generated by local sections $s \otimes s$ of $T^2(\Omega_{X/S})$, where s is a local section of $\Omega_{X/S}$ (only defined in some open set).

Define $\Omega^i_{X/S} = \bigwedge^i(\Omega_{X/S})$. Using the same terminology as in differential geometry to denote the equivalence class of elements in this algebra using wedge symbols, we have the differential map on local sections s of $\Omega^k_{X/S}$. If on an open set, $s = g_0 dg_1 \wedge \cdots \wedge dg_k$, then $ds := dg_0 \wedge dg_1 \wedge \cdots \wedge dg_k$. At the end, one gets the *de Rham complex*

$$\Omega^*_{X/S} : \mathcal{O}_{X/S} \xrightarrow{d} \Omega^1_{X/S} \xrightarrow{d} \Omega^2_{X/S} \to \cdots .$$

For properties of algebraic differential forms, see (EGA IV, §16.6 [3]). Then we arrive at the definition of algebraic de Rham cohomology:

Definition 2.2. The (relative) de Rham cohomology of X over S is the cohomology groups (sometimes called hypercohomology groups) over the de Rham complex

$$H^i_{dR}(X/S) := H^i R\Gamma(X, \Omega^*_{X/S}),$$

where $R\Gamma : \mathcal{D}(X, \pi^{-1}\mathcal{O}_S) \to \mathcal{D}(\mathcal{O}_S)$ is the derived functor from the derived category of sheaves of $\pi^{-1}\mathcal{O}_S$-modules to the derived category of sheaves of $\mathcal{O}_S$-modules.

The author does not have a clear picture of how to compute these cohomology groups currently (they involve taking total complexes), but a related keyword is the Hodge-to-de Rham spectral sequence.

Fact 1. Since π is proper, $H^n_{dR}(X/S)$ will be an algebraic vector bundle (locally free sheaf of finite rank). $H^n_{\mathrm{dR}}(X/S)$ is a vector bundle (locally free sheaf of finite rank) whose formation commutes with base change: if we have

$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

then there is a canonical isomorphism $H^n_{\mathrm{dR}}(X'/S') \xrightarrow{\sim} f^* H^n_{\mathrm{dR}}(X/S)$.

Fact 2 (Hodge Filtration). Derived from the more general Hodge filtration, we get a short exact sequence for a familly of elliptic curves over the base ring R ([7], A.1.2.1)

$$0 \to \underline{\omega}_{X/R} \to H^1_{dR}(X/R) \to H^1(X, \mathcal{O}_X) \simeq (\underline{\omega}_{X/R})^{\otimes -1} \to 0$$

and ([7], A.1) shows that there is a canonical but not functorial splitting (also known as the Hodge decomposition)

$$\underline{\omega}_{X/R} \oplus (\underline{\omega}_{X/R})^{-1} \simeq H^1_{dR}(X/R).$$

When $R = \mathbb{C}$, we recover the complex analytic Eicher-Shimura isomorphism

$$ES_k : S_k(\Gamma) \oplus \overline{S_k(\Gamma)} \to H^1_p(\Gamma, V_k(\mathbb{C})),\ (f, \bar{g}) \mapsto \phi_k(f) + \overline{\phi_k(g)},$$

where $\Gamma \subset SL_2(\mathbb{C})$ is a congruence subgroup, $S_k(\Gamma)$ is the vector space of cusp forms of weight k, H^1_p is the parabolic group cohomology, and $V_k(\mathbb{C})$ is isomorphic to the weight $k-2$ modular symbols. We won't pursue further in this direction, but for the purposes of this paper it suffices to know that $\underline{\omega}$ (the "holomorphic 1-forms") sits inside $H^1_{dR}(X/R)$ (all "closed deRham 1-forms").

3. Connections

We assume in this section that the base scheme S is over $\mathbb{C}$.

Definition 3.1 (Connection on vector bundles over a complex variety)**.** Let S be a smooth complex variety, E be a vector bundle on S. A connection ∇ on E is a $\mathbb{C}$-linear map

$$\nabla : E \to E \otimes_{\mathcal{O}_S} \Omega^1_{S/\mathbb{C}},$$

such that for local sections g of $\mathcal{O}_S$ and $\alpha \in E$,

$$\nabla(g\alpha) = g\nabla(\alpha) + \alpha \otimes dg.$$

By ([5], II.8.7), we can identify the global sections of the dual of the relative differentials

$$\Gamma(S, (\Omega^1_{S/\mathbb{C}})^\vee) = \Gamma(\mathrm{Hom}_{\mathbb{C}}(\Omega^1_{S/\mathbb{C}}, \mathbb{C})) \simeq \mathrm{Der}_{\mathbb{C}}(S, \mathbb{C}),$$

where the right hand side is the set of derivations $\{D : \mathcal{O}_S \to \Omega^1_{S/\mathbb{C}}\}$ which are maps of sheaves of abelian groups on S, which is a derivation of local rings at each point. So for each such derivation D, we can define a more geometric map which is the algebraic analogue of differentiating a vector field:

$$\nabla_D := (id_E \otimes D) \circ \nabla : E \to E$$

Theorem 3.1 (Gauss-Manin Connection, complex version)**.** *Let $\pi : X \to S$ be a smooth $\mathbb{C}$-morphism between smooth $\mathbb{C}$-schemes. Consider the corresponding map $\pi^{an} : X(\mathbb{C}) \to S(\mathbb{C})$ of complex manifolds. Then there is an identification $H^n_{dR}(X/S)^{an} = R^n\pi_*(\mathbb{Z}_{X(\mathbb{C})}) \otimes \mathcal{O}_{S(\mathbb{C})}$, where $\mathbb{Z}_{X(\mathbb{C})}$ is the constant sheaf of complex numbers on $X(\mathbb{C})$. There exists an integrable (*curvature$:= \nabla^2 = 0$*) connection, called the Gauss-Manin connection, whose incarnation in the complex form is given explicitly by*

$$\nabla : R^n\pi_*(\mathbb{Z}_{X(\mathbb{C})}) \otimes_{\mathbb{Z}_{S(\mathbb{C})}} \mathcal{O}_{S(\mathbb{C})} \to R^n\pi_*(\mathbb{Z}_{X(\mathbb{C})}) \otimes_{\mathbb{Z}_{S(\mathbb{C})}} \Omega^1_S \simeq H^n_{dR}(X/S)^{an} \otimes_{\mathcal{O}_S} \Omega^1_S,$$

$$\alpha \otimes f \mapsto \alpha \otimes df.$$

The verification of these claims is out of our scope, but our reference is Bloch's notes [1] and [2].

Theorem 3.2 (Gauss-Manin connection on the family $\pi : \mathbb{X} \to \mathcal{H}$)**.** *Let $E_2 : \mathcal{H} \longrightarrow \mathbb{C}$ be the weight 4 Eisenstein series:*

$$E_2(\tau) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n, \quad q = e^{2\pi i \tau},$$

and η be the global section of $H^1_{\mathrm{dR}}(\mathbb{X}/\mathcal{H})$ defined by

$$\eta(\tau) = \frac{1}{2\pi i} \wp_\tau(z) dz \in H^1_{\mathrm{dR}}(\mathbb{X}_\tau).$$

If $\nabla : H^1_{\mathrm{dR}}(\mathbb{X}/\mathcal{H}) \longrightarrow H^1_{\mathrm{dR}}(\mathbb{X}/\mathcal{H}) \otimes \Omega^1_{\mathbb{H}}$ denotes the Gauss-Manin connection, then on the canonical differential $\omega = dz$, we have

$$\nabla_D \omega = \eta - \frac{E_2}{12} \omega, \quad \nabla_D(\nabla_D \omega) = 0$$

where $D = \frac{1}{2\pi i} \frac{d}{d\tau}$.

Proof. We follow ([2], Lecture 2, Theorem 4.3). Under Grothendieck's de Rham theorem [4], we can work with singular cohomology by identifying the fibers of the bundle $(R^1\pi_*^{an}\mathbb{Z}_{\mathbb{X}})^\vee \to \mathcal{H}$ with $H_1(\mathbb{X}_\tau, \mathbb{Z}) \simeq \mathbb{Z} + \mathbb{Z}\tau$. The isomorphism also gives global sections (in other words, trivializations) (γ_1, γ_2) of $(R^1\pi_*^{an}\mathbb{Z}_{\mathbb{X}})^\vee$ such that $(\gamma_1(\tau), \gamma_2(\tau)) \mapsto (1, \tau) \in H_1(\mathbb{X}_\tau, \mathbb{Z})$. Then analytically, ([6],4.1.2) explains that the Gauss-Manin connection is determined by the formula

$$\int_{\gamma_i} \nabla_D(\omega) = D \int_{\gamma_i} \omega, \ i = 1, 2.$$

By Poincaré duality for analytic de Rham cohomology, it suffices to show that

$$\int_{\gamma_i} \nabla_D \omega = \int_{\gamma_i} \eta - \frac{E_2}{12} \int_{\gamma_i} \omega, \quad i = 1, 2.$$

By definition of $\omega = dz$, $\int_{\gamma_1} \omega = 2\pi i, \int_{\gamma_2} \omega = 2\pi i \tau$, so

$$\int_{\gamma_1} \nabla_D(\omega) = 0, \int_{\gamma_2} \nabla_D(\omega) = 1.$$

Thus, the statement is equivalent to showing the conditions

$$\int_0^1 \wp_\tau(z) dz = \frac{(2\pi i)^2}{12} E_2(\tau), \quad \int_0^\tau \wp_\tau(z) dz = 2\pi i + \frac{(2\pi i)^2}{12} E_2(\tau)\tau.$$

They hold by ([7], Lemma A1.3.9). $\square$

4. The Proof

Theorem 4.1 (Kodaira-Spencer, complex version)**.** *Let $\pi : X \to S$ be a family of elliptic curves with base scheme $\mathbb{C}$. Then the map $\underline{\omega}_{X/S}^{\otimes 2} \to \Omega_{S/\mathbb{C}}$*

$$s_1 \otimes s_2 \mapsto \langle s_1, \nabla s_2 \rangle_{dR}$$

is an isomorphism of sheaves of $\mathcal{O}_S$-modules.

Proof. Over $\mathbb{C}$, every family of elliptic curves is a pullback from the universal family $\pi : \mathbb{X} \to \mathcal{H}$ ([2], Lec 1, Thm 3.4), so it suffices to prove the statement for this family.

For this family, the first algebraic de Rham cohomology is identified the cohomology bundle over $\mathcal{H}$ with fiber isomorphic to the first de Rham cohomology $H^1(\mathbb{X}_\tau, \mathbb{R}) \simeq \mathbb{R}$, such that the canonical differential $\omega = d\tau$ trivializes the bundle. Then there exists an alternating and perfect *de Rham pairing* ([7], A1.3.17), analogous to the de Rham cup product, as

$$\langle , \rangle_{dR} : H^1_{dR}(\mathbb{X}/\mathcal{H}) \otimes H^1_{dR}(\mathbb{X}/\mathcal{H}) \to \mathcal{O}_{\mathcal{H}/\mathbb{C}}$$

that is characterized by $\langle dx/y, xdx/y \rangle_{dR} = 1$.

We compute the image of $\omega \otimes \omega$. By Theorem 3.2, we have $\langle \omega, \nabla_D \omega \rangle = \langle \omega, \eta \rangle - \langle \omega, \frac{E_2}{12}\omega \rangle$. The pairing is alternating, so the right hand side is equal to $d\tau$. It follows that $\langle \omega, \nabla\omega \rangle = 2\pi i d\tau$. As $d\tau$ generates the sheaf of differentials $\Omega_{S/\mathbb{C}}$, the Kodaira-Spencer map gives a surjective map between line bundles. The inverse is given by $d\tau \mapsto \frac{1}{2\pi i}\omega \otimes \omega$, so we have an isomorphism between invertible sheaves of $\mathcal{O}_{\mathcal{H}}$-modules. $\square$

References

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