MODULI STACK OF ELLIPTIC CURVES OVER $\mathbb{C}$

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Abstract. In this article, we construct and characterize the fine moduli stack of elliptic curves $\mathcal{M}_{1,1} \simeq [\mathfrak{h}/SL_2(\mathbb{Z})]$. This is the final project for Math 205A on Elliptic Curves at UCLA in Fall 2025 taught by Romyar Sharifi.

A recurring theme when studying the structure of algebraic curves belonging to a given family (for instance, smooth complete curves of genus g over a field k, or elliptic curves with a torsion subgroup of fixed order) is to study the parameter space of all these curves at once, and the resulting space is called a moduli space. Its existence is not unreasonable to expect: for the case of elliptic curves, the j-invariant gives a 1-1 correspondence between isomorphism classes of complex elliptic curves and $\mathbb{C}$, which can be compactified to the Riemann sphere $X(1) \simeq \mathbb{P}^1(\mathbb{C})$. In general, we can ask the question of how much information the moduli space ought to preserve from its parametrizing objects, and this is known as the *moduli problem*. It is not very clear a priori that this problem is well-posed, but a solution often lies between the extremes of coarse and fine moduli spaces. Coarse moduli spaces have the least structure, as for example, it can be shown that $X(1)$ is a coarse moduli space, and literally each elliptic curve on $X(1)$ is indistinguishable. Fine moduli spaces are more difficult to obtain, and it is known that no fine moduli space exists for complex elliptic curves.

We will specialize into the moduli problem of this case, and remedy the lack of a fine moduli space by introducing the theory of stacks originated from Grothendieck's algebraic geometry revolution. The main disadvantage of using stacks is its complicated definition and lost of concreteness, yet the approach is fruitful. For one reason, we would like to distinguish the "internal symmetries": let E be an elliptic curve over $\mathbb{C}$ with Weierstrass equation $y^2 = x^3 + ax + b,\ a, b \in \mathbb{C}$. For $j \neq 0, 1728$, there is a $\mathbb{Z}/2$-involution $(t, s) \mapsto (t, -s)$. Yet, when $a = 0$, we have a $\mathbb{Z}/6$-involution $(t, s) \mapsto (\zeta_3 t, -s)$, where $\zeta_3 = e^{\pi i/3}$, so the moduli stack ought to reflect these local behaviors.

Recall another way of constructing coarse moduli space of complex elliptic curves from complex analysis. Use the lattice representation $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$, where $\tau \in \mathfrak{h}$ and $\mathfrak{h}$ is the upper half plane, and quotient out its isomorphism classes which is characterized by the action of $SL_2(\mathbb{Z})$ by Möbius transformations, we obtain $\mathfrak{h}/SL_2(\mathbb{Z}) \simeq \mathbb{C}$. ([5], §I.2). This construction motivates our two main theorems. Due to space constraints, we will focus in the category of complex manifolds, but generalization to any base and higher genus curves using scheme theory is possible ([2], Appendix B).

Theorem 1. *The moduli stack of complex elliptic curves $\mathcal{M}_{1,1}$ is a complex analytic stack, isomorphic as stacks to $[\mathfrak{h}/SL(2, \mathbb{Z})]$.*

Theorem 2. *The exists a universal family of complex elliptic curves $\mathcal{U}_{1,1} \to \mathcal{M}_{1,1}$. Explicitly,*

$$\mathcal{U}_{1,1} \simeq [(\mathfrak{h} \times \mathbb{C})//(\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z}))].$$

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1. Stacks

1.1. **Prestacks.** The main paradigm shift in search for a solution for the moduli problem is to view the moduli space as the representating object of certain functors via Yoneda lemma. One example of this viewpoint is the *classifying space* of $S^1 \subset \mathbb{C}$, which is $\mathbb{CP}^\infty$. It gives an isomorphism

$$\mathrm{Htpy}(S^1, \mathbb{CP}^\infty) \simeq \{\text{complex line bundles over } S^1\}.$$

In other words, $\mathbb{CP}^\infty$ classifies all complex line bundles over S^1. In a similar vein, moduli stacks will classify families of elliptic curves over $\mathbb{C}$. Let $\mathbb{M}$ be the category of complex manifolds.

Definition 1 (Family of Elliptic Curves in $\mathbb{M}$)**.** A family of elliptic curves is a pair $(X \xrightarrow{\pi} U, e)$, where $X, U \in \mathbb{M}$, π is a smooth (surjective) fibration with section $e : U \to X$ such that for every $u \in U$, $(\pi^{-1}(U), e(u))$ is an elliptic curve. Namely, for every $p \in U$, there is a neighborhood V of p such that $\pi^{-1}(V) \simeq V \times \pi^{-1}(p)$ diffeomorphically, and this natural map respects basepoints.

We would very much like the notion of pull-back, for we can speak of universal objects.

Definition 2. The category $\mathcal{M}_{ell}$ of families of elliptic curves consists of families of elliptic curves as objects and pull-back squares as morphisms.

A vast generalization of the idea of families of geometric objects with a pull-back property is a prestack.

Definition 3 (Prestack on $\mathbb{M}$)**.** A prestack $\mathcal{X}$ on $\mathbb{M}$ is a 2-functor $\mathcal{X} : \mathbb{M}^{op} \to$ Groupoids. Namely, $\mathcal{X}(U)$ is a groupoid for every $U \in \mathbb{M}$, and for each holomorphic $f : U \to V$, we have a functor between groupoids $f^* : \mathcal{X}(V) \to \mathcal{X}(U)$, and we can speak about composing functors. There are a priori no interactions between compositions, so we impose the following conditions. If $f : R \to S$ and $g : S \to T$ are holomorphic, we require a natural isomorphism $(g \circ f)^* \simeq f^* \circ g^*$. Further, if $h : T \to U$ is another holomorphic map, then there exists a natural isomorphism $(g \circ f)^* \circ h^* \simeq f^* \circ (h \circ g)^*$.

Example 1 ($\mathcal{M}_{1,1}$)**.** *Define* $\mathcal{M}_{1,1} : \mathbb{M}^{op} \to$ Groupoids *by sending* $U \mapsto \pi^{-1}(U)$*, where* $\pi^{-1}(U)$ *is the category of family of elliptic curves on* U*.*

$\pi^{-1}(U)$ *is a groupoid, because we can always pull back* $X_1 \to U$ *and* $X_2 \to U$ *along the identity map, and swapping* X_1 *and* X_2 *gives a unique inverse map. For any* $f : U \to V$ *holomorphic, we have a functor* $\pi^{-1}(V) \to \pi^{-1}(U)$*, described by pulling back* $X_i \to U$ *along* f*, and we also get a unique map* $f^*(X_1) \to f^*(X_2)$ *by the universal property of pullbacks. One can check that it satisfies all axioms of a 2-functor, thus defines a prestack.*

Example 2 (Complex Manifolds)**.** *Fix* $U \in \mathbb{M}$*. We have a category* $\mathbb{M}_U$ *as follows. The objects are holomorphic maps* $f : V \to U$*, and morphisms are commutative triangles* $(V \xrightarrow{f} U) \xrightarrow{\phi} (W \xrightarrow{g} U)$*. We now assign any* $V \in \mathbb{M}$ *to the groupoid defined by: (i) objects are holomorphic maps* $f : V \to U$*, (ii) morphisms are* $\phi : (V \to U) \to (V \to U)$ *such that the forgetful functor applied to* $V \to V$ *is the identity. We call this the groupoid generated by the set* $\mathrm{Hom}_{\mathbb{M}}(V, U)$*. By defining precomposition as the action of functor on holomorphic maps, it defines a prestack which we denote by* $\underline{U}$*.*

We call this prestack $\underline{U}$ *analytic, for* U *represents some functor between stacks, to be made precise later.*

Definition 4 (Principal G-bundle)**.** Let G be a topological group, P and U complex manifolds. A principal G-bundle $\pi : P \to U$ is a smooth fibration with a continuous and freely transitive G-right action on P, such that for all $x \in U$, if $y \in \pi^{-1}(x)$, then $y \cdot g \in \pi^{-1}(x)$. Thus, the map $G \to \pi^{-1}(x)$, $g \mapsto yg$, is a homeomorphism.

Example 3 (Quotient (pre)stack)**.** *Let G be a complex Lie group acting biholomorphically on $M \in \mathbb{M}$. With notation as above, let $[M//G]$ be the category of pairs $(P \xrightarrow{\pi} U, P \xrightarrow{f} M)$, such that $P \to U$ is a principal G-bundle, and f is a G-equivariant analytic map. A morphism from $(P \xrightarrow{\pi} U, P \xrightarrow{f} M)$ to $(Q \xrightarrow{\sigma} V, Q \xrightarrow{g} M)$ is a G-equivariant map $\alpha : P \to Q$ and an analytic map $\psi : U \to V$ such that $\psi \circ \pi = \sigma \circ \alpha$ is Cartesian.*

Define a groupoid $[M//G](U)$ as the subcategory of principal G-bundles over U. As the bottom map $U \to U$ of the Cartesian square must be the identity, $P \to P$ will only permute the orbits at each fibre, so we see that the only morphisms in $[M//G](U)$ are actions of the elements of G. In other words, $\mathrm{Hom}_{[M//G](U)}(P, P)$ is bijective to G.

If $\psi : U \to V$ is holomorphic, then by pulling $Q \to V$ back along ψ, we get an assignment $[M//G](V) \to [M//G](U)$. One can further show this is functorial, and functor compositions respect the prestack axioms, so $[M//G] : U \mapsto [M//G](U)$ is a prestack.

By general abstract nonsense, we can define morphism of prestacks and the category of prestacks, denoted by $\mathcal{PS}$. Then, we have the following version of Yoneda ([3], 11.3.6),

Lemma 1 (Yoneda)**.** *Let $U, V \in \mathbb{M}$. Then*

$$\mathrm{Mor}_{\mathcal{PS}}(\underline{U}, \underline{V}) \simeq \mathrm{Hom}_{\mathbb{M}}(U, V).$$

1.2. **Stacks.** Prestacks are presheaves with respect to a Grothendieck topology on $\mathbb{M}$. A stack is thus a prestack with nice "gluing and descending" conditions. To simplify notation, denote $A|_U$ by the pullback of $A \in \mathcal{X}(V)$ by U, and $\phi|_U$ by the pullback of the morphism $\phi : A \to B$ in $\mathcal{X}(V)$ by a specified $f : U \to V$.

Definition 5 (Stack)**.** A stack $\mathcal{X} : \mathbb{M}^{op} \to$ Groupoids is a prestack such that for all $U \in \mathbb{M}$ and all open covering $(V_i)_{i \in I}$ of U, we have (c.f. the sheaf axioms)

(i) (Locality) If $A, B \in \mathcal{X}(U)$ and $(\phi_i : A|_{V_i} \to B|_{V_i})_{i \in I}$ is any family with $\phi_i|_{V_i \cap V_j} = \phi_j|_{V_i \cap V_j}$, then there is a unique morphism $\phi : A \to B$ such that $\phi|_{V_i} = \phi_i$.

(ii) (Gluing objects) Let $A_i \in \mathcal{X}(V_i)$ be a collection of objects, together with isomorphisms $\phi_{ij} : A^j_{V_i \cap V_j} \to A^i_{V_i \cap V_j}$ in $\mathcal{X}(V_{ij})$ such that the compositions of restriction are compatible, i.e. we have the cocycle condition $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ on $\mathcal{X}(V_{ijk})$ for any triple indices. Then there is an $A \in \mathcal{X}(U)$, together with isomorphisms $\phi_i : A|_{V_i} \to A_i$ such that $\phi_i \circ \phi_j^{-1} = \phi_{ij}$ in $\mathcal{X}(V_{ij})$.

Example 4. *The above three examples are all stacks. Note that the condition of local triviality in the family of elliptic curves is important for $\mathcal{M}_{1,1}$ to be a stack, see ([3], 11.6.5.)*

Consider now the category $\mathcal{S}$ of stacks. To define the moduli problem, we set out to define what an analytic stack means.

Definition 6 (Pullback of stacks)**.** If $\mathcal{Y} \to \mathcal{X}$ and $\mathcal{Z} \to \mathcal{X}$ are two stack morphisms, then the pullback $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}$ is a stack satisfying a 2-commutative pullback square.

Definition 7 (Representable stack morphism)**.** A representable stack morphism $\mathcal{M} \to \mathcal{X}$ is a morphism such that for every $V \in \mathbb{M}$, there exists $U_V \in \mathbb{M}$ such that

$$\mathcal{M} \times_{\mathcal{X}} \underline{V} \simeq \underline{U_V}.$$

In other words, to study any morphism from a complex manifold V into $\mathcal{X}$, we can study the base change by $\mathcal{M} \to \mathcal{X}$ to a complex analytic map $U \to V$ via the Yoneda lemma.

$$\begin{tikzcd}
\mathcal{M} \times_{\mathcal{X}} \underline{V} \simeq \underline{U} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\
\underline{V} \arrow[r] & \mathcal{X}
\end{tikzcd}$$

Example 5. *The space $[\bullet/G]$ is called the classifying stack of G, and denoted as BG in literature. It is important as it classifies all principal G bundles in the sense that*

$$\mathrm{Hom}_{\mathcal{S}}(S, BG) \simeq \{\textit{isomorphism classes of } G\textit{-bundles on } S\}.$$

Then the canonical morphism $\bullet \to BG$ is representable, because one can show that $\underline{V} \times_{BG} \bullet$ is naturally isomorphic to $\underline{P}$ coming from a principal G-bundle.

Definition 8 (Submersion, surjectivity)**.** A representable stack morphism $F : \mathcal{M} \to \mathcal{X}$ is called a submersion if every induced complex analytic morphism from definition 7 is a submersion. F is said to be surjective if for all $S \in \mathbb{M}$, $P \in \mathcal{X}(S)$, there exists an open cover $(S_i)_{i \in I}$ of S and $P_i \in \mathcal{M}(S_i)$ for each $i \in I$, such that $F_{S_i}(P_i) \simeq P\mid_{S_i}$.

Intuitively, one may see surjectivity as an analogue of local triviality of a fibration on stacks. We are ready to define the moduli problem:

Definition 9 (Analytic Stack)**.** An analytic stack is a stack $\mathcal{X}$ on $\mathbb{M}$ equipped with a $X \in \mathbb{M}$ and a stack map $p : \underline{X} \to \mathcal{X}$, such that p is a representable surjective submersion.

Definition 10 (Coarse Moduli Space)**.** Let $\mathcal{X}$ be an analytic stack. A coarse moduli space for $\mathcal{X}$ is a pair (M, π), $M \in \mathbb{M}$, $\pi : \underline{M} \to \mathcal{X}$ a morphism of stacks, such that for all $\phi : X \to \underline{N}$, there exists a unique morphism of stacks $\phi' : \underline{M} \to \underline{N}$ satisfying $\phi' \circ \pi = \phi$. Notice that uniqueness of the coarse moduli space is automatic.

Example 6. *If $M \in \mathbb{M}$ induces an analytic stack, G is a complex Lie group acting on the right on M, then $[M//G]$ is an orbifold ([3], 11.4.10), so it is an analytic stack. Then the canonical map $\underline{M} \to [M//G]$ makes M into the coarse moduli space of $[M//G]$ ([3], 11.5.2). Therefore, the existence of coarse moduli space is equivalent to the existence of a categorical quotient for an action G on M. This holds, for example, in the class of GIT quotients in geometric invariant theory.*

In fact, it can be shown that the general moduli stack $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ can be shown to admit a coarse moduli space $\bar{M}_g$ by a projective scheme. ([1], Theorem A.)

Definition 11 (Fine Moduli Space)**.** Let $\mathcal{X}$ be an analytic stack. A fine moduli space for $\mathcal{X}$ is a complex manifold M such that $\mathcal{X} \simeq \underline{M}$. We say $\mathcal{X}$ is representable.

Example 7. *By ([3], 11.1.14), $\mathcal{M}_{1,1}$ cannot be represented by a manifold.*

Definition 12 (Stacky version of family of elliptic curves)**.** Let $\mathcal{X}$ be a stack over $\mathbb{M}$. A family of elliptic curves over $\mathcal{X}$ is a morphism of stacks $p : \mathcal{E} \to \mathcal{N}$ such that p is representable and for all $S \in \mathbb{M}$ and all $f : \underline{S} \to \mathcal{X}$, the pullback $f^*\mathcal{E} = S \times_{\mathcal{X}} \mathcal{E} \in \mathbb{M}$ is a family of elliptic curves over S in the complex manifold sense.

Definition 13 (Universal family)**.** Let $p : \mathcal{E} \to \mathcal{X}$ be a family of elliptic curves. It is said to be universal if for all $U \in \mathbb{M}$ and all $X \in \mathcal{X}(U)$, there exists a unique morphism of stacks $f : X \to \mathcal{X}$ such that $f^*\mathcal{E} \simeq \underline{U}$.

Definition 14 (The Moduli Problem for $\mathcal{M}_{1,1}$)**.** To solve the moduli problem in our context means to find a universal family $p : \mathcal{S}_{1,1} \to \mathcal{M}_{1,1}$. This will justify the terminology of calling $\mathcal{M}_{1,1}$ the moduli stack because it classifies families of elliptic curves.

2. The Moduli Problem for $\mathcal{M}_{1,1}$

In this chapter, we will give an exposition of proofs of theorems 1 and 2 following [3]. The strategy is to first study elliptic curves with a framing, because we can get rid of the non-trivial automorphisms. In particular, an explicit fine moduli space for framed elliptic curves exists, and we can verify it easily on the level of manifolds and pass it back to stacks. This is why this approach is so rewarding.

2.1. **The Moduli Problem for Framed elliptic curves.**

Definition 15 (Framed elliptic curves)**.** Let (E, e) be a complex elliptic curve. A framing of (E, e) is a basis $([a], [b])$ of the free $\mathbb{Z}$-module $H_1(E, \mathbb{Z}) = \pi_1(E, e) = \mathbb{Z}^2$, with the additional property that for every non-zero holomorphic 1-form ω of E,

$$\frac{\int_a \omega}{\int_b \omega} \text{ has positive imaginary part.}$$

Proposition 1. $\mathrm{Aut}(E, e, ([a], [b])) = \{id\}$. $\mathfrak{h}$ *represents isomorphism classes of framed elliptic curves, called the Teichmüller space.*

Proof. E represents a framed lattice $\Lambda = a\mathbb{Z} \oplus b\mathbb{Z}$ such that b/a has positive imaginary part. Two lattices are isomorphic if and only if there exists $u \in \mathbb{C}^*$ with $a = ua'$, $b = ub'$, so the map

$$E \mapsto \frac{\int_a \omega}{\int_b \omega} \in \mathfrak{h}$$

is bijective. This also shows $(E, e, ([1], [\tau])) \simeq (E, e, ([1], [\tau']))$ iff $\tau \neq \tau'$ (ruling $\tau = -\tau'$ by positivity), i.e. distinct framed lattices are not homothetic. □

Lemma 2. *Given two framed elliptic curves* $(E_i, e_i, (a_i, b_i))$*,* $i = 0, 1$*, there exists a orientation preserving homoeomorphism* $f : (E_0, e_0) \to (E_1, e_1)$ *such that* $f_*(a_0) = a_1$ *and* $f_*(b_0) = b_1$*.*

Proof. Write them as the form $(\mathbb{C}/\Lambda_i, 0, (1, \tau_i))$ for some $\tau_i \in \mathbb{C}$ with $\mathrm{im}(\tau_i) > 0$. One checks that the map $\tilde{f} : \mathbb{C} \to \mathbb{C}$,

$$z \mapsto \frac{(\tau_1 - \bar{\tau}_0)z - (\tau_1 - \tau_0)\bar{z}}{\tau_0 - \bar{\tau}_0}$$

has positive $\mathbb{R}$-determinant, and descends to an orientation preserving homeomorphism on the elliptic curves. □

By the lemma, we can fix a framing (a_0, b_0) of (E_0, e_0), then other framings are given by the image of an orientation preserving homeomorphism. This shows the bijection

$$\{\text{framings of } (E_0, e_0)\} \leftrightarrow \mathrm{Mod}(E_0, e_0) = \mathrm{Homeo}^+(E_0, e_0)/\,\mathrm{Homotopy}\,.$$

Furthermore, the lemma implies that $\mathrm{Mod}(E_0, e_0)$ acts free and transitively on $H_1(E, \mathbb{Z}) \simeq \mathbb{Z}^2$, so we have

$$\mathrm{Mod}(E_0, e_0) \simeq SL(2, \mathbb{Z}).$$

Definition 16 (Framed elliptic curves)**.** Fix (E_0, e_0). To describe a framing of elliptic curve on (E, e), it suffices to specify the data

$$[f] \in \mathrm{Homeo}^+((E_0, e_0), (E, e))/\mathrm{Homotopy}.$$

Denote $\mathrm{Fr}_{(E_0,e_0)}(E, e) \simeq \mathrm{Homeo}^+((E_0, e_0), (E, e))/\mathrm{Homotopy}$ as the set of all framings of (E, e). There is a freely transitive action by the lemma of $\mathrm{Mod}(E_0, e_0)$ on $\mathrm{Fr}_{(E_0,e_0)}(E, e)$ through $[f]\cdot[g] = [f \circ g]$.

To define family of framed elliptic curves, we have to present a continuous way of choosing a framing. It is achieved by defining a frame bundle:

Definition 17 (Frame bundle)**.** As a family of elliptic curves $(E \xrightarrow{\pi} U, e)$ in $\mathbb{M}$ is a locally trivial topological bundle, we can construct the frame bundle

$$\mathrm{Fr}(E, e) := \coprod_{u \in U} \mathrm{Fr}_{(E_0,e_0)}(E_u, e(u)) \to U$$

where $\mathrm{Fr}_{(E_0,e_0)}(E_u, e(u)) = \mathrm{Homeo}^+((E_0, e_0), (E_u, e(u)))/\mathrm{Homotopy}$.

Definition 18 (Family of framed elliptic curves)**.** Fix an elliptic curve (E_0, e_0). A family of framed elliptic curves is a quadruple (E, π, e, σ) where $\pi : E \to U$ is a family of elliptic curves and $\sigma : U \to \mathrm{Fr}(E, e)$ is a continuous section.

As we defined $\mathrm{Fr}(E, e)$ fibre-wise, by local triviality, we can make sense of morphisms between frame bundles. Thus a morphism of family of framed elliptic curves is just a morphism of family of elliptic curves with a frame bundle map. Further, if $E \to X$ is a framed family of elliptic curves and $f : Y \to X$ is a holomorphic map, then f induces a natural frame $f^* \mathrm{Fr}(E, e) = \mathrm{Fr}(f^*E, f^*e)$. Thus we can form the category of framed elliptic curves and category of framed elliptic curves over a smooth manifold U. Proceeding similarly, we obtain a stack (proof omitted).

Proposition 2. *The functor $\mathcal{M}_{1,1}^{fr} : U \mapsto$ (groupoid of framed elliptic curves over U) is a stack.*

Theorem 3 (Universal family for $\mathcal{M}_{1,1}^{fr}$)**.** *There exists a universal family $\pi : \mathcal{U}_{1,1}^{fr} \to \mathfrak{h}$.*

Proof. Let $\mathbb{Z}^2$ act on $\mathfrak{h} \times \mathbb{C}$ from the right as follows. If $(m, n) \in \mathbb{Z}^2$ and $(\tau, z) \in \mathfrak{h} \times \mathbb{C}$, then

$$(\tau, z) \cdot (m, n) = (\tau, z + m\tau + n).$$

Define the quotient manifold $\mathcal{U}_{1,1}^{fr} = (\mathfrak{h} \times \mathbb{C})/\mathbb{Z}^2$, and a holomorphic map $\pi : \mathcal{U}_{1,1}^{fr} \to \mathfrak{h}$ by projection onto the first component. By Yoneda, it suffices to check that this is a universal family in the category of family of framed elliptic curves on $\mathfrak{h}$.

For every $\tau \in \mathfrak{h}$, the preimage $\pi^{-1}(\tau)$ is isomorphic to an elliptic curve $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ by the identity map. Also, the map $\tau \mapsto (\tau, 0)$ defines a canonical section for this map, so π is a family of elliptic curves. Finally, we choose a framing through the orientation preserving homeomorphism:

$$\sigma(\tau) : \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}) \to \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z}),$$

setting $\tau_0 = i$ and $\tau_1 = \tau$ in Lemma 2. By construction, we have

$$(\mathcal{U}_{1,1}^{fr}\mid_{\{\tau\}}, e(\tau), \sigma(\tau)) \simeq (\mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z}), 0, (1,\tau));$$

so indeed, if we have another framed family of elliptic curves $V \to \mathfrak{h}$, then it must factor through π. This proves that π is universal. $\square$

Theorem 4 (Fine moduli space for $\mathcal{M}_{1,1}^{fr}$). *The stack $\mathcal{M}_{1,1}^{fr}$ is representable and is isomorphic to $\underline{\mathfrak{h}}$.*

Proof. For every $U \in \mathbb{M}$, $\mathcal{M}_{1,1}^{fr}(U)$ is the category of families of framed elliptic curves parametrized by U. For each $(E \to U, e, \sigma)$ an object in this category, we have an isomorphism

$$(E_s, e(s), \sigma(s)) \to (\mathbb{C}/\Lambda_{(1,\tau_s)}, 0, (1,\tau_s))$$

for each $s \in U$. Since this isomorphism is a part of the more general equivalence of categories between framed elliptic curves and framed lattices, this isomorphism varies smoothly on s[1]. Hence, the assignment $s \mapsto \tau_s$ defines a functor

$$\mathcal{M}_{1,1}^{fr}(U) \to \underline{\mathfrak{h}}(U).$$

More generally, we can check it defines a stack morphism $\mathcal{M}_{1,1}^{fr} \to \underline{\mathfrak{h}}$.

To construct the inverse, we have the pullback map on the level of manifolds,

$$\underline{\mathfrak{h}}(U) = \mathrm{Mor}(U, \mathfrak{h}) \to \mathcal{M}_{1,1}^{fr}(U), \quad f \mapsto f^*(\mathcal{U}_{1,1}^{fr}).$$

By putting a framing on each fiber through $f : U \to \mathfrak{h}$, we have a morphism in the category of family of framed elliptic curves, and it induces on the stack level an inverse stack morphism $\underline{h} \to \mathcal{M}_{1,1}^{fr}$. As the action of $SL_2(\mathbb{Z}) \simeq \mathrm{Mod}(E_0, e_0)$ is encoded in the framing, we have to check that this stack isomorphism is $SL_2(\mathbb{Z})$-equivariant, which is done in [3]. $\square$

2.2. **The Moduli Problem for Elliptic curves.**

A rough proof of Theorem 1. Let $U \in \mathbb{M}$. Then an object in $\mathcal{M}_{1,1}(U)$ is a family $(E, \pi : E \to U, e)$ of elliptic cuvres parametrized by U. We can construct a holomorphic map $\mathrm{Fr}(E, e) \to \mathfrak{h}$ as follows.

Let $[f] \in \mathrm{Fr}(E, e)$ be an element, which by definition gives the homotopy class of an oriented homeomorphism

$$f : (\mathbb{C}/\Lambda_{(1,i)}, 0, ([a_0], [b_0])) \to (E_{\pi([f])}, e(p([f]))),$$

where $p : \mathrm{Fr}(E, e) \to U$ is the canonical projection. Then we can define u as

$$[f] \mapsto \tau_{p([f])},$$

where

$$\tau_s = \frac{\int_{b_0} \sigma(s)^* \omega_s}{\int_{a_0} \sigma(s)^* \omega(s)} \in \mathfrak{h}.$$

[1]A completely rigorous argument is in ([3], pp.416-417).

We can similarly check that u is $SL(2,\mathbb{Z})$ equivariant, and by construction $\mathrm{Fr}(E,e) \to U$ is a principal $SL(2,\mathbb{Z})$ bundle. Hence, $(\mathrm{Fr}(E,e) \to U, u : \mathrm{Fr}(E,e) \to S) \in [\mathfrak{h}//SL(2,\mathbb{Z})]$.

Conversely, consider $(P \to S, u : P \to \mathfrak{h})$, where $P \to S$ is a principal $SL(2,\mathbb{Z})$ bundle. By pulling back the universal family of framed elliptic curves by u, we get a manifold $\hat{E}$

$$\begin{array}{ccc} \hat{E} := u^*\mathcal{U}_{1,1}^{fr} & \longrightarrow & \mathcal{U}_{1,1}^{fr} \\ \big\downarrow & & \big\downarrow \\ \underline{P} & \longrightarrow & \mathfrak{h} \end{array} ;$$

Lemma 3. *$SL(2,\mathbb{Z})$ acts on $\mathcal{U}_{1,1}^{fr}$ free and properly.*

The proof of the lemma is contained in the proof of theorem 2 below, in the construction of the universal family of $\mathcal{M}_{1,1}$. With the lemma, $E = \hat{E}/SL(2,\mathbb{Z})$ is a quotient manifold over $S = P/SL(2,\mathbb{Z})$, so indeed $E \in \mathcal{M}_{1,1}(S)$. One can check this defines an inverse. $\square$

Proof of Theorem 2. $\mathcal{U}_{1,1}^{fr}$ is given by the quotient of $\mathfrak{h} \times \mathbb{C}$ by the $\mathbb{Z}^2$ action. We extend it to a $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$ action on $\mathfrak{h}$ as follows. The group law of $\hat{G} := \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$ is

$$\begin{aligned} &\left((m,n), \begin{pmatrix} a & c \\ b & d \end{pmatrix}\right)\left((m',n'), \begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix}\right) \\ &= \left((m,n) + (am'+cn', bm'+dn'), \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a' & c' \\ b' & d' \end{pmatrix}\right), \end{aligned}$$

then the right action of $\hat{G}$ on $\mathfrak{h} \times \mathbb{C}$ is

$$(\tau, z) \cdot \left((m,n), \begin{pmatrix} a & c \\ b & d \end{pmatrix}\right) := \left(\frac{a\tau+b}{c\tau+d}, \frac{z+m\tau+n}{c\tau+d}\right).$$

This action induces a $SL(2,\mathbb{Z})$-action on $u^*\mathcal{U}_{1,1}^{fr}$, and is free and proper because the $SL(2,\mathbb{Z})$-action is free and proper on P. Summarizing, we have a $SL(2,\mathbb{Z})$-equivariant map $u^*\mathcal{U}_{1,1}^{fr}(U) \to P$, and a $SL(2,\mathbb{Z})$-principal bundle $u^*\mathcal{U}_{1,1}^{fr}(U) \to U$, so by ([4], Proposition 3.5.8, p.156), the quotient stack $\mathcal{U}_{1,1} = [\mathcal{U}_{1,1}^{fr}//SL(2,\mathbb{Z})]$ is a universal family. $\square$

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