INTRODUCTION TO GEOMETRIC MOTIVIC INTEGRATION

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Abstract. We explain the construction of geometric motivic integration in algebraic geometry. This is the final project for Math 245A Real Analysis, taught by Terence Tao in Fall 2025.

Motivic Integration is a tool in birational geometry proposed by Kontsevich [12] in 1995 to solve the problem of invariance of Hodge structures for crepant solutions of Calabi-Yau varieties with mild singularities. It was first formulated geometrically. Subsequently, Denef, Loeser, Batyrev, and Looijenga developed the foundations of the subject.

Typically, we start with a complex projective variety X with some mild singularities allowed. We can associate to X a domain of integration, the arc space $\mathcal{J}_\infty(X)$ which contains the $\mathbb{C}[[x]]$ points of X. On this space, we can define a class of "elementary sets¨ which are called stable sets. We can then define a premeasure μ, which takes values in a very large and still poorly understood ring $\hat{\mathcal{M}}_k$ that contains much geometric information of the "universal Euler characteristic¨. The most important class of measurable functions is the contact order of an arc along a divisor, and we will define their motivic integrals taking values in $\hat{\mathcal{M}}$. This integral obeys a nice "change of variables formula¨, and for some class of divisors, there is a useful formula that makes computation feasible in practice. Last but not least, it produces a birational invariant allowing Kontsevich to prove the birational invariance of Hodge structures.

We shall see despite the value ring deviating sharply from standard measure theory, it still makes sense to talk about the motivic measure μ as a classical premeasure in a Boolean algebra of measurable sets in the arc-space. The following table taken from [2] provides a comparison.

	Lesbeques	motivic
space	$\mathbb{R}^n$	$\mathcal{J}_\infty(\mathbb{A}^n)$
values of measure	$\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$	$K_0(\mathrm{Var}_k) \subseteq \mathcal{M}_k \subseteq \hat{\mathcal{M}}_k$
cubes around point a	$\{x \in \mathbb{R}^n \mid \lVert x - a \rVert \leq 1/m\}$	$\{\gamma \in \mathcal{J}_\infty(\mathbb{A}^n) \mid \lVert \gamma - a \rVert \leq 1/m\}$
measurable set	σ–algebra generated by cubes	algebra of stable/measurable sets
volume of cube	$(2/m)^n$	$(\mathbb{L}^{-m})^n$
transformation rule	$\int_A h(f) df = \int_{g^{-1}(A)} h(f(x)) \operatorname{Jac}(f) dx$	$\int_A \mathbb{L}^{-\operatorname{ord}_D} = \int_{f_\infty^{-1}(A)} \mathbb{L}^{-\operatorname{ord}_D + K_{X'/X}}$

The main references of this article are [2] and [17], although their articles refer to hundreds of papers which we are not able to fully cite. Their surveys came out in the 2000s, so this article might not contain state-of-the-art results, but we only intend to give an exposition to the ideas used here. The resulting article is entirely in my own words. I wish to thank Terence Tao for his encouragement to work on this project.

Date: Fall 2025.

1. Space of Formal Arcs

Assume X is a scheme of finite type over an algebraically closed and characteristic zero field k of dimension n.

Definition 1 (Space of m-jets)**.** Let $m \geq 0$ be an integer. An m-jet of X is a $k[t]/t^{m+1}$-point of X, namely a morphism $\theta : \operatorname{Spec} k[t]/t^{m+1} \to X$. It can be shown that the collection of m-jets form a scheme $\mathcal{J}_m(X)$, called the m-th jet scheme, or space of truncated arcs, which represents the contravariant functor $Z \mapsto \operatorname{Hom}(Z \times \operatorname{Spec} k[t]/t^{m+1}, X)$. In other words, the $k[t]/t^{m+1}$-points of X are in bijection with the k-points of $\mathcal{J}_m(X)$:

$$\operatorname{Hom}(Z \times \operatorname{Spec} k[t]/t^{m+1}, X) = \operatorname{Hom}(Z, \mathcal{J}_m(X)).$$

From this definition it is immediately clear that $\mathcal{J}_0(X) = X$. We further define the truncation morphisms $\pi^m_{m-1} : \mathcal{J}_m(X) \to \mathcal{J}_{m-1}(X)$ by pulling back the canonical map $\operatorname{Spec} k[t]/t^m \to \operatorname{Spec} k[t]/t^{m+1}$, and $\pi^m : \mathcal{J}_m(X) \to \mathcal{J}_0(X)$ as the map induced by $\operatorname{Spec} k \to \operatorname{Spec} k[t]/t^{m+1}$.

Example 1. *$X = \mathbb{A}^d_{\mathbb{C}}$. $\mathcal{J}_m(X) = \{(a^{(1)}(t), \cdots, a^{(d)}(t)) \mid a^{(i)}(t) \in \mathbb{C}[t]/t^{m+1}\}$. Each $a^{(i)}(t)$ is a truncated power series $a^{(i)}_0 + a^{(i)}_1 t + \cdots + a^{(i)}_m t^m$, where $a^{(i)}_j \in \mathbb{C}$. The m-th jet scheme is thus isomorphic to $\mathbb{A}^{(m+1)d}_{\mathbb{C}}$.*

Example 2. *$X = \{y^2 = x^3\} \subset \mathbb{A}^2_{\mathbb{C}}$. Then $\mathcal{J}_1(X) = \{(a_0 + a_1 t, b_0 + b_1 t) \mid (b_0 + b_1 t)^2 = (a_0 + a_1 t)^3 \bmod t^2\} = \{(a_0 + a_1 t, b_0 + b_1 t) \mid a_0^3 = b_0^2,\ 2b_0 b_1 = 3a_0^2 a_1\}$. The truncation map $\pi^1 : \mathcal{J}_1(X) \to X$ then is the projection $(a_0 + a_1 t, b_0 + b_1 t) \mapsto (a_0, b_0)$. The fibre at a non-zero point (a_0, b_0) is the tangent line to X at (a_0, b_0). We can thus identify $\mathcal{J}_1(X) \simeq TX$ the tangent bundle at X. This holds more generally for any X (see [9], Ex. II.2.8).*

Example 3. *Continuing the example above, we can see that $\pi^3_2 : \mathcal{J}_3(X) \to \mathcal{J}_2(X)$ is not surjective. This is due to the singularity at $(0,0)$.*

Patching all data together, we obtain the space of arcs, a variety of infinite dimension over k.

Definition 2 (Space of Arcs)**.** The arc space $J_\infty(X)$ is the inverse limit of $J_m(X)$. Concretely, it is constructed by

$$\varprojlim \operatorname{Hom}(\mathcal{O}_X, k[t]/t^{m+1}) \simeq \operatorname{Hom}(\mathcal{O}_X, \varprojlim k[t]/t^{m+1}) \simeq \operatorname{Hom}(\mathcal{O}_X, k[[t]]).$$

We can also define the m-th truncation $\pi_m : \mathcal{J}_\infty(X) \to \mathcal{J}_m(X)$.

Example 4. *$X = \{y^2 = x^3\}$, then $\mathcal{J}_\infty(X) = \{(\sum a_n t^n, \sum b_n t^n) \mid (\sum b_n t^n)^2 - (\sum a_n t^n)^3 = 0\}$, an infinite dimensional variety with lots of relations.*

Kolchin proved that if X is irreducible, then $\mathcal{J}_m(X)$ is irreducible. The following proposition describes for smooth k-schemes the fibre bundle property (compare to the example $\{y^2 = x^3\}$ above).

Proposition 1 ([2], Prop 2.3)**.** *Let X be a smooth k-scheme of dimension n, then $\mathcal{J}_m(X)$ is locally an $\mathbb{A}^{nm}$-bundle over X. In particular, $\mathcal{J}_m(X)$ is smooth of dimension $n(m+1)$ and $\mathcal{J}_{m+1}(X)$ is locally an $\mathbb{A}^n$-bundle over $\mathcal{J}_m(X)$.*

2. Measure on the Space of Formal Arcs

2.1. **Cylinders and Stable sets.** The constructible subsets of a scheme are built out of finite unions and intersections of closed and open sets. This defines a convenient class of topologically simple subsets that we can stratify the scheme with, and they also arise naturally (for example, the image $\{x \neq 0\} \cup \{0, 0\}$ of $\mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}$ under the map $(x, y) \mapsto (x, xy)$ is constructible but not closed or open). It would be nice to hope for a measure on the family of constructible subsets.

Definition 3 (Constructible subsets of a scheme)**.** Let Y be a scheme. The space of constructible subsets is the smallest boolean algebra of sets containing the closed subsets of Y in the Zariski topology.

For example, when Y is a complex projective variety (or Noetherian scheme), the constructible subsets are the finite disjoint union of locally closed subvarieties.

Definition 4 (Cylinders/Constructible subsets of $\mathcal{J}_\infty(X)$)**.** A subset $A \subset \mathcal{J}_\infty(X)$ is said to be cylindrical or constructible if $A = \pi_m^{-1}(C)$ for some $m \geq 0$ and $C \subset \mathcal{J}_m(X)$ constructible.

The prototypes of measurable sets are the stable sets.

Definition 5 (Stable sets)**.** A subset $A \subset \mathcal{J}_\infty(X)$ is said to be stable if there exists $M >> 0$, such that $\pi_m(A)$ is constructible for all $m \geq M$, $A = \pi_m^{-1}(A_m)$ and $\pi_m^{m+1} : A_{m+1} \to A_m$ is a locally trivial $\mathbb{A}^n$ bundle.

All stable sets are constructible. When X is smooth, constructibility is equivalent to stability.

Lemma 1 ([5])**.** *If $A \subset \mathcal{J}(X)$ is constructible, and $A \cap \mathcal{J}(X_{\text{sing}}) = \varnothing$, then A is stable.*

2.2. **The ring of values.** Having defined the prototypes of measurable sets in the motivic measure, we describe its ring of values. The fundamental relation of this ring is the ability to break up a variety into a closed subvariety and its complement, and declaring their "sum" to be the same. This relation is called the scissor relations (Hales [8] has a very nice exposition).

Definition 6 (Grothendieck ring of Var_k)**.** Let k be any field, Var_k be the category of varieties over k. The Grothendieck ring of Var_k is the ring

$$K_0(\mathrm{Var_k}) = \{\text{isomorphism classes of finite type varieties over } k\}/([X] = [Y] + [X - Y])$$

where $Y \subset X$ is a closed k-subvariety. Multiplication is given by taking fiber products over k. To be consistent with literature, we denote $\mathbb{L}$ as the class of $\mathbb{A}^1_k$, and 1 as the class of $\operatorname{Spec} k$.

One should be aware that this definition of K_0 does not fit into the general definition of K_0 of an exact category, which relies on the existence of exact sequences. It is very hard to study the structure of $K(\mathrm{Var}_k)$. Poonen (2002) [16] proved that it is not a domain. Bot-Cangini-van Santen (2025) [3] showed using supersingular elliptic curves that if k has characteristic equal to 11 or at least 17, then a closely related ring has nilpotent elements.

Example 5. $[\mathbb{P}^n] = \mathbb{L}^n + \mathbb{L}^{n-1} + \cdots + \mathbb{L} + 1.$

Example 6. *If A is a constructible subset of X, then it makes sense to talk about its class in $K_0(\mathrm{Var}_k)$, and we have a map $A \mapsto [A]$ from the boolean algebra of constructible subsets of X to $K_0(\mathrm{Var}_k)$.*

Example 7 (Important example)**.** *Let $f : Y \to X$ be a piecewise trivial fibration, i.e. $X = \coprod_{finite} X_i$, $f^{-1}(X_i) \simeq X_i \times Z$ for some Z, and f is a projection into each X_i. Then $[Y] = [X] \cdot [Z]$ by the definition of fiber products.*

Now let X be nonsingular of pure dimension d, $A = \pi_m^{-1}(C)$ where C is some contructible set in $\mathcal{J}_m(X)$. Then we claim that for all $n \geq m$, $[\pi_n(A)] = \mathbb{L}^{(n-m)d}[C]$. This is because by Proposition 1, the map $\pi_m^n : \mathcal{J}_n(X) \to \mathcal{J}_m(X)$ is a locally trivial fibration, by Proposition 1.

It follows that the quantity $\frac{[\pi_n(A)]}{\mathbb{L}^{nd}}$ are all equal for $n \geq m$ in the localized ring $K_0(\mathrm{Var}_k)[\mathbb{L}^{-1}]$. This justifies the terminology of calling A stable sets. We would want to enlarge K_0 slightly to accomodate this, and it turns out that localizing at $\mathbb{L}$ is enough for motivic integration (geometrically, $\mathbb{L}$ is the trivial vector bundle). We denote this ring as $\mathcal{M}_k$.

We might be content to define the *naïve motivic measure*, for any stable sets A,

$$\mu_X(A) = \lim_{n\to\infty} \frac{[\pi_n(A)]}{\mathbb{L}^{nd}}.$$

In practice, however, this definition is too restrictive, because we want to include the more general class of all sets $A = \pi_m^{-1}(C)$ where C is constructible, but the limit might not exist in $\mathcal{M}_k$ for the non-smooth case.

Example 8. *Assuming X is nonsingular of pure dimension d, then $\mathcal{J}_\infty(X) = \pi_m^{-1}(\mathcal{J}_m(X))$ and $\mathcal{J}_m(X)$ is a locally an $\mathbb{A}^{md}$-bundle over $\pi_0(X)$. This means that $\frac{[\mathcal{J}_m(X)]}{\mathbb{L}^{md}} = \frac{[X]\cdot[\mathbb{L}^{md}]}{[\mathbb{L}^{md}]} = [X]$, so $\mu_X(\mathcal{J}_\infty(X)) = [X]$.*

Example 9. *Consider $X = \{xy = 0\} \subset \mathbb{A}^2_{\mathbb{C}}$, then it is not smooth at the origin. We claim that*

$$\lim_{n\to\infty} \frac{[\pi_n(A)]}{\mathbb{L}^{nd}} = \lim_{n\to\infty} \frac{2\mathbb{L}^{n+1}-1}{\mathbb{L}^n} = \lim_{n\to\infty} 2\mathbb{L} - \frac{1}{\mathbb{L}^n} \notin \mathcal{M}_k.$$

It suffices to show that $[Y_n] = [\pi_n(\mathcal{J}_\infty(X))] = 2\mathbb{L}^{n+1} - 1$. This is an illuminating exercise that also shows $\pi_m(\mathcal{J}_\infty(X)) \subsetneq \mathcal{J}_m(X)$ in general. We know that $\mathcal{J}_\infty(X) = \{(\sum a_i t^i, \sum b_i t^i) \mid (\sum a_n t^n) \equiv 0$ or $(\sum b_n t^n) \equiv 0\}$, so by truncating at t^{n+1} we easily see that the part $\pi_n(\mathcal{J}_\infty(X)) \cap \{\sum a_n t^n = 0\}) \simeq \mathbb{C}^{n+1}$ and similarly for the other chart, and they overlap only at the origin, so $[Y_n] = 2\mathbb{L}^{n+1} - 1$. Note that in contrast, $\mathcal{J}_n(X) = \{(\sum a_i t^i, \sum b_i t^i) \bmod t^{n+1} \mid (\sum a_n t^n)(\sum b_n t^n) = 0 \bmod t^{n+1}\} = \{(\sum a_i t^i, \sum b_i t^i) \bmod t^{n+1} \mid \sum_{i+j=k, i\neq j} a_i b_j = 0,\ 0 \leq k \leq n\}$, which is much larger.

Although the limit does not exist in the above example, we would still expect the measure of $\mathcal{J}_\infty(X)$ to be $2\mathbb{L}$. To make limit well-defined, we replace $\mathcal{M}_k$ by its completion with respect to a natural absolute value.

Definition 7 (Value Ring for Motivic Integration)**.** We can complete $\mathcal{M}_k$ with respect to a natural norm as follows. Define the dimension map $\dim : \mathcal{M}_k \to \mathbb{Z} \cup \{-\infty\}$, where $\tau \in \mathcal{M}_k$ is d-dimensional if $\tau = \sum a_i[X_i]$ where X_i is dimension $\leq d$, and if we cannot write another decomposition of dimension all $\leq d-1$. Define $\dim[\varnothing] = -\infty$. Compose dim with the exponential map gives a map $\|\cdot\|_k : \mathcal{M}_k \to \mathbb{R}^{\geq 0}$. It is a non-archimedean norm:

(i) $\|A\| = 0$ iff $A = 0 = [\varnothing]$, (ii) $\|A + B\| \leq \max\{\|A\|, \|B\|\}$, (iii) $\|A \cdot B\| \leq \|A\|\|B\|$.

Hence, completing $\mathcal{M}_k$ with respect to this norm gives us a strong notion of convergence. In particular, $\sum_{n=1}^\infty a_n$ converges if and only if $a_n \to 0$. We call this ring $\hat{\mathcal{M}}_k$.

2.3. **The Measure.** Having defined the ring which we would expect μ_X to extend to all constructible sets, we now make rigorous the class of measurable sets. In spirit, it is very similar to approximating Lebesgue measurable sets by elementary sets; intuitively, we can think of stable sets as "dense¨ in the constructible sets.

Definition 8 (Measurable Sets)**.** A subset $C \subset \mathcal{J}_\infty(X)$ is called measurable if for all $n \in \mathbb{N}$ there is a stable set C_n and stable sets $D_{n,i}$ for $i \in \mathbb{N}$ such that

$$C \triangle C_n \subset \bigcup_{i \in \mathbb{N}} D_{n,i} \quad \text{and} \quad \dim \mu(D_{n,i}) \leq -n \, \forall i.$$

Because we will focus on the smooth case, we will not prove this proposition.

Proposition 2. *Any cylinder $C \subset \mathcal{J}_\infty(X)$ is measurable.*

Definition 9 (Motivic Measure on Measurable Sets)**.** Let $C \subset \mathcal{J}_\infty(X)$ be measurable. We define the motivic measure as

$$\mu_X(C) = \lim_{n \to \infty} \mu_X(C_n) \in \hat{\mathcal{M}}_k.$$

Proposition 3. *The limit exists in the above definition. The collection of measurable sets form a Boolean algebra, and μ_X is σ-additive on disjoint unions. In other words, μ_X is a measure.*

Proof. The existence of limits is in ([2], pp.22). $\varnothing$ is by convention measurable. Suppose C is measurable. Choose C_n and $D_{n,i}$ as above. Then $C^c \triangle C_n^c = C \triangle C_n$, and each of the C_n^c are stable sets, so C^c is measurable. A similar proof shows that finite unions of measurable sets are measurable, so this collection is a Boolean algebra. It is not hard to see that μ_X is σ-additive on stable sets, so approximating countable unions with limits of countable union of stable sets proves σ-additivity. $\square$

3. Motivic Integration of an Effective Divisor

From an algebra of measurable sets and a measure μ_X, we can define the notion of integration. We will restrict our attention to X being smooth over k algebraically closed from now on.

We can describe an important class of measurable functions, analogus to the simple functions, as follows.

Definition 10 (Order of Y in X)**.** Let $Y \subset X$ be a closed subscheme of X with $\dim Y < \dim X$, defined by a sheaf of ideals $I_Y \subset \mathcal{O}_X$. The order of Y in X is the function

$$\mathrm{ord}_Y : \mathcal{J}_\infty(X) \to \mathbb{N} \cup \{\infty\},$$

sending $\theta : \mathcal{O}_X \to k[[t]]$ to the order of vanishing of θ along Y, namely, to the maximum e of ideals (t^e) such that $\theta(I_Y) \subset (t^e)$. In the literature, they are also called $k[t]$-simple functions.

The order obeys some important properties.

Proposition 4. *i) Each fiber* $\mathrm{ord}_Y^{-1}(n)$ *is a cylinder, for all* $n \in \mathbb{N} \cup \{\infty\}$. *In other words,* ord_Y *is a measurable function, agreeing with classical terminology.*

ii) $\mathrm{ord}_Y^{-1}(\infty)$ *has measure zero.*

Proof. Put the definition of $\mathrm{ord}_Y(\theta)$ in another way, it is the largest e such that the image of I_Y in the truncation map $\pi_{e-1} \circ \theta$ vanishes. This detects the deviation of a jet in $\mathcal{J}_m(X)$ from belonging to $\mathcal{J}_n(Y)$. We thus have

$$\begin{aligned}
\mathrm{ord}_Y(\theta) \neq 0 &\iff \pi(\theta) \in Y, \\
\mathrm{ord}_Y(\theta) \geq s &\iff \pi_{s-1}(\theta) \in \mathcal{J}_{s-1}(Y) \text{ and} \\
\mathrm{ord}_Y(\theta) = \infty &\iff \theta \in \mathcal{J}_\infty(Y).
\end{aligned}$$

This translates to $\mathrm{ord}_Y^{-1}(\geq s) = \pi_{s-1}^{-1}(\mathcal{J}_{s-1}(Y))$, which is a cylinder, so each fibers is a difference of cylinders $\mathrm{ord}_Y^{-1}(\geq s) - \mathrm{ord}_Y^{-1}(\geq s+1)$, also a cylinder. When $s = 0$, we have $\mathrm{ord}_Y^{-1}(0) = \pi^{-1}(X) - \pi^{-1}(Y)$, so we get part (i).

To prove (ii), it suffices to prove that $\mathcal{J}_\infty(Y) = \bigcap \mathrm{ord}_Y^{-1}(\geq s)$ has measure zero, assuming $\dim Y < \dim X$. The case when Y is singular is harder (though not very hard) to prove, so we focus on the non-singular case. Suppose Y has dimension $n-c$. Then for the smooth case, $\dim \mathcal{J}_s(Y) = (n-c)(s+1)$. By example 7, the measure of $\pi^{-1}(\mathcal{J}_s(Y))$ is $[\mathcal{J}_s(Y)] \cdot \mathbb{L}^{-ns}$. Therefore, the dimension of $\pi^{-1}(\mathcal{J}_s(Y))$ is $(n-c)(s+1) - ns = n - cs - c \to -\infty$ as $s \to \infty$. Therefore, the measure of the intersection $\bigcap_{1 \leq s \leq n} \mathrm{ord}_Y^{-1}(\geq s)$ converges to zero in the non-archimedean norm, giving the claim by downward monotone convergence theorem (!). $\square$

Definition 11 (A class of integrable functions and its integral)**.** Let $A \subset \mathcal{J}_\infty(X)$ be constructible and $\alpha : A \to \mathbb{Z} \cup \{\infty\}$ a function such that the conclusions of Proposition 4 holds. Then it is measurable, and we define the motivic integral of this function (denoted $\mathbb{L}^{-\alpha}$) as

$$\int_{\mathcal{J}_\infty(X)} \mathbb{L}^{-\alpha} d\mu_X = \sum_{s=0}^{\infty} \mu(\alpha^{-1}(s)) \mathbb{L}^{-s}.$$

Typically, Y is defined as the variety to the surface defined by an effective divisor.

Definition 12 (Order associated to an effective Cartier divisor)**.** Let D be a subvariety of X locally given by one equation (D is a Cartier divisor). Then, define $\mathrm{ord}_t D$ as mapping $\gamma \in \mathcal{J}_\infty(X)$ to $\mathrm{ord}\, f_D(\gamma)$, where f_D is a local equation near $\pi_0(\gamma)$. It is an order, so it is measurable.

Example 10. *Let* Y *be a smooth divisor in* X, *i.e. a smooth closed subscheme of pure codimension* 1. *By proposition* 1, $\mathcal{J}_s(Y)$ *is locally an* $\mathbb{A}^{(n-1)s}$*-bundle over* Y. *We have, for each level sets* $s \geq 1$,

$$\mathrm{ord}^{-1}(s) = \pi^{-1}(\mathcal{J}_s(Y)) - \pi^{-1}(\mathcal{J}_{s-1}(Y)),$$

which has measure, by the computation in proposition 4,

$$[\mathcal{J}_{s-1}(Y)] \cdot \mathbb{L}^{-n(s-1)} - [\mathcal{J}_s(Y)] \cdot \mathbb{L}^{-ns} = [Y] \cdot \mathbb{L}^{(n-1)(s-1)} \cdot \mathbb{L}^{-n(s-1)} - [Y] \cdot \mathbb{L}^{(n-1)s} \cdot \mathbb{L}^{-ns} = [Y](\mathbb{L}-1)\mathbb{L}^{-s}.$$

Therefore, its associated integral is

$$\begin{aligned}
\int_{\mathcal{J}_\infty(X)} \mathbb{L}^{-\mathrm{ord}_Y} d\mu_X &= [X - Y] + [Y] \sum_{s=1}^{\infty} (\mathbb{L} - 1)\mathbb{L}^{-s} \cdot \mathbb{L}^{-s} = [X - Y] + [Y]\mathbb{L}^{-2} \sum \mathbb{L}^{-2s} \\
&= [X - Y] + [Y]\mathbb{L}^{-2} \frac{1}{1 - \mathbb{L}^2} = [X - Y] + [Y](\mathbb{L} + 1)^{-1} = [X - Y] + [Y][\mathbb{P}]^{-1}.
\end{aligned}$$

There is a convenient formula to compute a large class of nonsingular effective divisors, first proved by Batyrev.

Theorem 1 ([4]). *Let X be nonsingular and $D = \sum_{i \in S} N_i D_i$ a normal crossings divisor on X, i.e. all D_i are nonsingular hypersurfaces intersecting transversely (and occurring with multiplicity N_i). Denote $D_I^\circ := (\cap_{i\in I} D_i) \setminus (\cup_{\ell \notin I} D_\ell)$ for $I \subset S$; the $D_I^\circ, I \subset S$, form a natural locally closed stratification of X (note that $D_\emptyset^\circ = X \setminus (\cup_{\ell \in S} D_\ell)$). Then*

$$\int_{\mathcal{J}(X)} \mathbb{L}^{-\operatorname{ord}_t D} d\mu = \sum_{I \subset S} [D_I^\circ] \prod_{i \in I} \frac{\mathbb{L} - 1}{\mathbb{L}^{1+N_i} - 1}.$$

Motivic integration theory obeys a change of variables formula, proven by Kontsevich.

Theorem 2 (Transformation Rule, [12]). *Let $f : X' \to X$ be a proper birational morphism of smooth k-schemes (k characteristic 0) and let D be an effective divisor on X. Then*

$$\int_{\mathcal{J}_\infty(X)} \mathbb{L}^{-\operatorname{ord}_D} d\mu_X = \int_{\mathcal{J}_\infty(X')} \mathbb{L}^{-\operatorname{ord}_{f^{-1}D + K_{X'/X}}} d\mu_{X'},$$

where $K_{X'/X}$ is the relative canonical divisor.

A proof can be found in [2], involving a detailed analysis in the case of blowing up a smooth subvariety. Then one can use the weak factorization theorem (which is actually a very deep result) of Abramovich-Karu-Matsuki-Wlodarczyk [1] below to factor any birational map as a sequence of blowup maps to prove the general case. We will not show it here because the author is not competent enough to demonstrate a full proof.

Theorem 3 (Weak Factorization). *Let $\varphi : X' \to X$ be a birational map between smooth complete varieties over k of characteristic zero. Then φ can be factored as*

$$\begin{tikzcd}
& X_1 \arrow[dl] \arrow[dr] & & X_3 \arrow[dl] \arrow[dr] & & \arrow[dl] \cdots & & \arrow[dr] & X_{n-2} \arrow[dl] \arrow[dr] & & X_n \arrow[dl] \arrow[dr] & \\
X' & & X_2 & & X_4 & \cdots & & & X_{n-3} & X_{n-1} & & X
\end{tikzcd}$$

such that all indicated maps are a blowup at a smooth center. Furthermore, there is an index i such that the rational maps to X to the left ($X' \dashleftarrow X_j$ for $j < i$) and the rational maps to X to the right ($X_j \dashrightarrow X$ for $j > i$) of that index are in fact regular maps.

This completes the essential construction of the motivic measure for most applications.

4. (Incomplete) list of Applications

In this last section, we survey some applications of motivic integration in literature to illustrate its influence.

4.1. **Kontsevich's Theorem.**

We can understand Kontsevich's original proof of the birational invariance of Hodge structures in the following version of the theorem. Let M be a Calabi-Yau manifold of dimension d, i.e. a nonsingular, compact (more generally, complete, but they are equivalent over $\mathbb{C}$) variety which admits a nowhere vanishing top regular differential form ω_M. Alternatively, the canonical divisor K_M of M is zero.

Theorem 4 (Kontsevich, [12]). *Let X and Y be birationally equivalent Calabi-Yau manifolds. Then $[X] = [Y]$ in $\hat{\mathcal{M}}_{\mathbb{C}}$.*

Proof. The proof leverages the transformation rule of motivic integration. As X and Y are birationally equivalent non-singular Calabi-Yau, there exist a nonsingular complete variety Z and birational morphisms $h_X : Z \to X$, $h_Y : Z \to Y$. (This result relies on Hironaka's Theorem of resolution of singularities, see for example 3.16 for a proof.) By the transformation rule, we have

$$[X] = \mu_X(\mathcal{J}_\infty(X)) = \int_{\mathcal{J}_\infty(X)} 1 d\mu_X = \int_{\mathcal{J}(Z)} \mathbb{L}^{-\operatorname{ord}_t K_{Z|X}} d\mu_X = \int_{\mathcal{J}(Z)} \mathbb{L}^{-\operatorname{ord}_t K_Z} d\mu_X,$$

because the canonical divisor of X is trivial. This computation is the same for $[Y]$, so $[X] = [Y]$. □

Corollary 1. *Birationally equivalent Calabi-Yau manifolds have the same Hodge-Deligne polynomial, so they have the same Hodge numbers.*

Proof. For nonsingular manifolds, the universal Euler-Characteristic specializes to the Hodge-Deligne polynomial, so they induce maps $\mathcal{M}_{\mathbb{C}} \to \mathbb{Z}[[u, v]]$. Composing with the motivic measure gives the result. □

A more general formulation using essentially the same proof involving crepant resolutions can be easily found in the literature, which will not be stated here.

4.2. **Batyrev's McKay Correspondence.**

Recall that the McKay Correspondence states that the representation-theoretic data of finite subgroups of $SU_2(\mathbb{C})$ gives the same information as the data of exceptional divisors through blowing up the singularities of its associated variety, classified using Dynkin quivers. We state a motivic version of the McKay Correspondence conjectured by Reid, proved by Batyrev, and developed by Denef and Loeser.

Theorem 5 ([6], Theorem 3.6, Corollary 5.1-5.3). *Let $d \geq 1$ be an integer and let k be field of characteristic 0 containing all d-th roots of unity. Let G be a finite subgroup of $SL_n(k)$ of order d. Let $h : Y \to X$ be a crepant resolution of $X = \mathbb{A}^n_k/G$. Then its exceptional divisor can be characterized as*

$$[h^{-1}(0)] = \sum_{[\gamma] \in \operatorname{Conj}(G)} \mathbb{L}^{n-\omega(\gamma)}.$$

4.3. **More applications.**

- A motivic analog of Igusa's Rationality Theorem [11] (1975) of the Poincare series. It is proven by Denef and Loeser in 1999 [5].
- A main conjecture towards this direction is the Monodromy conjecture, relating the coefficients of topological zeta function to a system of eigenvalues. The author supposes that this area is interesting because it is akin to the modularity theorem in number theory, one version stating the solutions modulo p of an elliptic curve with integral coefficients as we vary p counts arises as coefficients of an eigenform. The modularity theorem is covered in [7].
- Arithmetic Motivic Intgeration. Hrushovski and Kazhdan developed using model theory a theory of integration over valued fields [10].
- Topological mirror symmetry test for singular Calabi-Yau varieties, with a very recent result developed by Lee, Lian, and Yau (2025)[13].
- Stringy invariants for varieties with log-terminal singularities. For an exposition, see ([2], Section 5.)
- Birational invariant for varieties of non-negative Kodaira dimension, assuming the minimal model program. A set of course notes can be found in [14].

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